Original Article

“A Study Of Fixed Point Theory In Banach Spaces”

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DOI: 10.47750/pnr.2023.14.03.190

Abstract

In this paper we list some geometric properties of Banach spaces and some fixed point theory results which we use in the subsequent topic . Certain geometric properties of a Banach space X are helpful to study the existence of fixed points for non expansive mappings and asymptotically non expansive mappings on closed bounded convex subsets of X. In this paper first we give a brief introduction to Geometry of Banach spaces, a brief introduction to fixed point theory, a brief introduction to best proximity point theorems, and finally we give contents of the paper.

Keywords - Fixed point; Uniform convexity; Uniform convexity in every direction; Asymptotic center; Property UC; k-strongly Chebyshev; k-strongly convex; ; Best proximity point.

INTRODUCTION –

Browder [10] and Göhde [13] proved the existence of fixed points for non expansive mappings on weakly compact convex subsets of a uniformly convex Banach space. In 1965, Kirk [23] proved the same result for weakly compact convex subsets of a Banach space that have normal structure. Goebel and Kirk introduced a generalized notion of non expansive mappings, called asymptotically non expansive mappings, in [24]. They proved that every asymptotically non expansive map on a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Recently Ran and Rearing’s [23] have developed a new direction in fixed point theory. They proved a fixed point theorem, which is an analogue of the classical Banach contraction theorem, for continuous maps on a complete metric space with partial order relation wherein contraction condition is satisfied on comparable elements. After this, many authors [17, 18, 19, 20] have studied results of this type in different directions.

Bin Dehairs and Khamis [6] proved an analogue of the fixed point theorem of Browder [10] and Göhde [25] for monotone non expansive mappings. Recently, Alfredian and Khamis[1] have proved an analogue of Goebel and Kirk’s fixed point theorem [24] for monotone asymptotically non expansive maps in a partially ordered uniformly convex Banach space. Edelstein [14] introduced the concept of asymptotic center and proved some fixed point theorems for generalized non expansive mappings. Edelstein’s result not only prove the existence of a fixed point but also the existence of a fixed point having a preassigned location. Using the concept of asymptotic center we obtain the existence of fixed points having preassigned location for a wider class of asymptotically non expansive map-pings in a uniformly convex Banach space. From this generalization, Alfredian and Khamsi’s [1] result for continuous maps can be recovered. We observe that some fixed point theorems proved in [6, 1] for partially ordered sets hold for unsoldered sets as well. We may recall that a unsoldered set is a set equipped with a quayside relation which is a binary relation satisfying reflexive and transitive properties. We obtain some results for order-preserving continuous maps without mono-tone non expansive condition in a uniformly convex Banach space having monotone norm. Under similar situation we also obtain a fixed point theorem in a reflexive Banach lattice. Throughout this chapter, a Banach space is over K which is either the filed R of real numbers or the filed C of complex numbers unless otherwise specified.

The norm on a Banach space is denoted by k · k. In this paper we prove that if every closed convex subset of X is k-strongly Chebyshev then X is k-strongly convex. It is also known that a strongly convex space X is characterized in terms of property UC (see Theorem 1). In the previous theorem we introduced a property called property k-UC as a generalization of property UC. By replacing property UC by property k-UC in, we prove that X is k-strongly convex.

In our results we use a generalized version of Cantor intersection theorem (Theorem 3). To prove this result we need the following Theorem 3, which was proved by Kirk.

ISHIKAWA ITERATIONS IN BANACH SPACES –

Let X be a Banach space and C be a non empty closed convex subset of X. Let T:C→C be such that, IITx - Ty|| < a, II X - y|| +a21| x - Tx|| + as ||y - Ty|| + a4||x - Ty|| + asH y -Tx|| (i) for all x, y e C , where $a_j > 0$, i;f=i a; < 1 .

Hardy and Rogers , Rhoades , Wong , have obtained the fixed point ofthe operators which satisfies the condition (i) under different hypothesis. If 2f=i ^i < 1» then the method of iteration is very helpful to determine the fixed point. On the other hand if Xi o.i is allowed to become equal to one, then the method of iteration fails and the problem of finding the fixed point becomes non routine. In this case to obtain the fixed point, one has to restrict the space or impose certain additional conditions on the operator We denote X and X' a real Banach space and the dual space of X, respectively.

Let F be a mapping of a nonempty set X into itself. Then F is said to have a fixed point if there exists a point u in X such that Fu = u. By a fixed point theorem we shall understand a statement which asserts that under certain conditions on the mapping F and on the space X, a mapping F of X into itself admits one or more fixed points. Be The first theorem of this kind was obtained by Brouwer in 1912, which states that every continuous self mapping of a closed ball of R" has a fixed point. This result was not of much use because of the finite dimension of the concerned space. The result was extended to infinite dimensional spaces by Birkhoff and Kellogg [9] in 1922 proving that any compact convex set in the spaces L^[0,1] and C[0,1] hasfixed point property for continuous functions. In the same year Banach [8] obtained a fundamental and constructive fixed point result, known as Banach contraction theorem, which states that a contraction mapping of a complete metric space into itself has a unique fixed point, where a contraction mapping F is a function from a metric space (X,d) into itself satisfying d(F_x, F_y) < a d(x, y) for all x, y in X and 0 < a < 1.

In 1930, Schauder [146] established the resuh on convex subset of a normed linear space, stating that "Let F be a continuous mapping of a non-empty convex subset K of a normed linear space X into a compact subset of K, then F has'a fixed point in K" .The fixed point theory in general and of Banach in particular having varied applications in non linear analysis, attracted mathematicians for its further development, which actually led to many generalizations of the contraction mappings theorem during sixties and afterwards. In 1961

Definition 1 : Opial's condition [21] A Banach space X is said to satisfy Opial's condition if for any sequence {X_n} in E, X_n →x implies that $\lim_{n\to\infty} sup$||X_n-x| | < $\lim_{n\to\infty} sup$ || X_n- y || for all y ∈ X with xi∈y.

The object of this chapter is to prove that if X is a uniformly convex Banach space which satisfies Opal's condition, C is a non empty closed convex subset of X and T :C →*C which satisfies (i) with F(T) # 4> then for any initial Xi in C the iterates defined by with { α_{2n} } and { β_{2n}} are chosen so that α_{2n} ∈ [a,b] some a, b with 0 < a < b < 1 converges weakly to a fixed point of T.

Further we prove that if X is strictly convex Banach space and T(C) is contained in a compact subset of C, then the iterates defined by (ii), where { α_{2n}} and { β_{2n}} are chosen so that a2n G [a,b], β_{2n}, G [0,b] for some a, b with 0 < a < b < 1 converges strongly to a fixed point of T. We now prove the weak convergence theorem which is connected with the result of Takahashi and Kim [26]

Definition 2 : Two selfmaps A and S of a metric space (X, d) are called R-weakly commuting at a point x in X if d(ASx, SAx) < R d[Ax,Sx) for some R > 0.

Definition 3: The maps A and S are called pointwise R-weakly commuting on X if given x in X there exists R > 0 such that d(ASx,SAx) <R_d(Ax,Sx)

Theorem 1: Let C be a T - invariant subset of a normed linear space X. Let T: C → C be a contractive operator on C and let x ∈ F(T).

If D e X is nonempty, compact and star shaped, then D ∩F (T) ≠ ∅

Further Hicks and Humphries [56] have shown that if the assumption T: C → C can be weakened to the condition T : C → C if y ∈C i.e. y ∈ D is not necessary in the interior of C, where δC denotes the boundary of C .

Theorem 2 : Let X be a Banach space. Let T, I: X → X be operators and C be a subset of X such that T : δC → C and x ∈ F(T) ∩F(I). Fiuther suppose that T and I satisfy IITx – Tyll < a |)Ix - lyll + (1 - a) max {||Ty - Ty||. ||Tx - Ix||, V2 (||Ty - Iy|| + ||Tx - Ix) **Proof**: Let y ∈ Da and hence ly is in Da, since I (Da) = Da. Further if y ∈ δC then Ty is in C since T(δC) ∈ C From (i), it follows that IITy-xll = II Ty – Txll < a II ly - 1 x|| + (1 - a) max {||Ty - Iy||, ||Tx - 1 x||, Vi (|| Ty - Iy|| +1| Tx - Ix ||}<a||Iy.x| | + (l-a)(||Ty-x| | + ||Iy-x||) or, a||Ty-x| |< ||Iy.x| | = l|TIx,,-Tx|| < a II l,, - Ixll + (1 - a) max {|| Tlxn - l'x,,||. || Tx - Ix||, V2 (||TIXn - l'x,,|| + Tx - Ix ||} which implies, as n —^ a II Iw - x|| < a II Iw - x||. Hence Iw = x By (i) again, we have

||Tw-x|| = ||Tw-Tx|| < a 11 Iw - Ixjl + (I - a) max (jj Tw - Iwjj, jj Tjc - Ixjj, V2 (||Tw - Iw|| + jjTx - Ix|l)} which gives,
||Tw-x||<(l-a)||Tw-x| |.

Theorem 3: [15] Let K be a nonempty closed convex subset of a uniformly convex Banach space. Let T : K → K be a map such that (T^nx) is bounded for some x ∈ K. If c is the asymptotic center of (T^nx) with respect to K and there exists N such that kT c – T^nx k ≤ kc − T^nx −1xk for n ≥ N, then c is a fixed point of T.

We generalize the above theorem of Edelstein [15] involving a finite number of closed convex subsets instead of a single set.

Example: Let us consider the bounded sequence x_n := $(-1)^n$(0, 1) in R 2 with sup norm and let K := {(x, 0) : 0 ≤ x ≤ 1}. Note that k(0, 1) − (0, x)K_∞ = 1 and k(0, −1) − (x, 0)K_∞ = 1.

for every 0 ≤ x ≤ 1. Hence $\lim_{n\to\infty}$ sup k(x, 0) − x_nk = 1 for every (x, 0) in K. Then the asymptotic center of the sequence (x_n) with respect to the closed convex set K.
However, under some suitable assumptions on the set K and on the space X we see that the asymptotic center is singleton, as the following two lemmas show.

Lemma 1.1: [1] Let K be nonempty closed convex subset of a uniformly convex Banach space X. Then the asymptotic center of every bounded sequence in X with respect to K is singleton. Further, if (X_n) is a bounded sequence in X with asymptotic center x0 with respect to K and if (X_n) is a sequence in K such that r(Y_m,(X_n)) → r(x_0,(X_n)) as m → ∞, then $y_m \to y_0$.
The second part of the above Lemma 1.1 is also derived from Corollary-1 in[25].
Similarly if the space X is uniformly convex in every direction then the asymp-totic center of a bounded sequence is singleton with respect to every weakly com- pact convex set.

Lemma 1.2: [25] Let K be a nonempty weakly compact convex subset of a Ba-nach space X which is uniformly convex in every direction. Then the asymptotic center of every bounded sequence in X with respect to K is singleton. 26 Edelstein [15] proved the following fixed point theorem for generalized none x- pensive mappings in a uniformly convex Banach space. Also this theorem shows that the asymptotic center of an iterated sequence is a fixed point of a non expansive mapping. This result helps to locate the fixed point of a non expansive mapping.

Remark: Theorem-1 of Edelstein is a special case of Theorem- 2 by taking A1 = A2 = · · · = Ak. In the statement of Theorem 2, we used Lemma 1.2 for asserting that the asymptotic center for (T^n x_0) is singleton. Analogously, making use of Lemma we obtain the following theorem.

REFERENCES:

1. Alfredian, M. R. and M. A. Khamis (2018). A fixed point theorem for monotone asymptotically non expansive mappings. Proc. Amer. Math. Soc.,146(6), 2451–2456.
2. Alspach, D. E. (1981). A fixed point free non expansive map. Proc. Amer.Math. Soc., 82(3), 423–424.
3. Ayerbe Toledano, J. M., T. Domínguez Benavides, and G. López Acedo, Measures of noncompactness in metric fixed point theory, volume 99 of Op-erator Theory: Advances and Applications. Burkhouse Verlag, Basel, 1997.
4. Bachar, M. and M. A. Khamis (2015). Fixed points of monotone mappings and application to integral equations. Fixed Point Theory Appl., 2015:110, 7.
5. Bernal, J. and F. Sullivan (1983). Multidimensional volumes, super-reflexivity and normal structure in Banach spaces. Illinois J. Math., 27(3), 501–513.
6. Bin Dehaish, B. A. and M. A. Khamis (2016). Browder and Göhde fixed point theorem for monotone non expansive mappings. Fixed Point Theory Appl., Paper No. 20, 9.
7. Birkhoff, G., Lattice theory. Third edition. American Mathematical Society Colloquium Publications, Vol. XXV. American Mathematical Society, Provi- dence, R.I., 1967.
8. Boyd, D. W. and J. S. W. Wong (1969). On nonlinear contractions. Proc. Amer. Math. Soc., 20, 458–464.
9. Brodskiĭ, M. S. and D. P. Mil0man (1948). On the center of a convex set. Doklady Akad. Nauk SSSR (N.S.), 59, 837–840.
10. Browder, F. E. (1965). Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. U.S.A., 54, 1041–1044.
11. Caristi, J. (1976). Fixed point theorems for mappings satisfying inwardness conditions. Trans. Amer. Math. Soc., 215, 241–251.
12. Clarkson, J. A. (1936). Uniformly convex spaces. Trans. Amer. Math. Soc., 40(3), 396–414.
13. Downing, D. and W. A. Kirk (1977). Fixed point theorems for set-valued mappings in metric and Banach spaces. Math. Japon., 22(1), 99–112.
14. Edelstein, M. (1972). The construction of an asymptotic center with a fixed- point property. Bull. Amer. Math. Soc., 78, 206–208.
15. Edelstein, M. (1974). Fixed point theorems in uniformly convex Banach spaces. Proc. Amer. Math. Soc., 44, 369–374.
16. Eldred, A. A., W. A. Kirk, and P. Veeramani (2005). Proximal normal structure and relatively nonexpansive mappings. Studia Math., 171(3), 283–293.
17. Eldred, A. A., V. S. Raj, and P. Veeramani (2011). On best proximity pair theorems for relatively u-continuous mappings. Nonlinear Anal., 74(12), 3870–3875.
18. Eldred, A. A. and P. Veeramani (2006). Existence and convergence of best proximity points. J. Math. Anal. Appl., 323(2), 1001–1006.
19. Espínola, R. (2008). A new approach to relatively nonexpansive mappings. Proc. Amer. Math. Soc., 136(6), 1987–1995.
20. Espínola, R. and A. Fernández-León (2011). On best proximity points in metric and Banach spaces. Canad. J. Math., 63(3), 533–550.
21. Fan, K. (1969). Extensions of two fixed point theorems of F. E. Browder. Math. Z., 112, 234–240.
22. Fernández-León, A. (2010). Existence and uniqueness of best proximity points in geodesic metric spaces. Nonlinear Anal., 73(4), 915–921.
23. Garkavi, A. L. (1962). On the optimal net and best cross-section of a set in a normed space. Izv. Akad. Nauk SSSR Ser. Mat., 26, 87–106.
24. Goebel, K. and W. A. Kirk (1972). A fixed point theorem for asymptotically non expansive mappings. Proc. Amer. Math. Soc., 35, 171–174.
25. Goebel, K. and W. A. Kirk, Topics in metric fixed point theory, volume 28 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
26. Göhde, D. (1965). Zum Prinzip der kontraktiven Abbildung. Math. Nachr., 30, 251–258.
27. Jachymski, J. (2008). The contraction principle for mappings on a metric space with a graph. Proc. Amer. Math. Soc., 136(4), 1359–1373.