Original Article

IMPATIENT CUSTOMERS IN A MARKOVIAN QUEUE WITH MULTIPLE WORKING VACATION AND SERVER BREAKDOWN

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Abstract

We look at a single server Markovian queue with multiple Working vacation and impatient consumers in this research. We determine the distributions for the mean queue length and mean waiting time, as well as their stochastic decomposition structures, using the matrix geometric technique. Finally, we provide numerical data to highlight the effects of system characteristics on performance measurements.

Keywords: M/M/1 queue, Working vacation, Stochastic decomposition, Matrix-geometric solution.

1. Introduction

Under the vacation strategy, Queuing Models have been centered broadly in terms of their application in PCs, PC frameworks, and board construction. Various vacation policies increase the flexibility for the best planning of Queuing frameworks. First focused on M/M/1 Queuing models with varied working vacations (WV)[11]. Recently, a GI/M/1 bunch service line with a high WV approach was released [6]. To break down the M/M/c line, a single WV was employed by [5]. In addition, he gave a quick rundown of current improvements in vacation queueing models by [4].

Previously, writers such as [1], [12], and [13] considered queuing models with customer impatience, where the cause of concern was either a massive delay previously known about the line, or a significant wait expected by a customer upon arrival. They concentrated on the Single Server Markovian Queuing System with WV and Vacation Interruptions with Setup Time, and later expanded their work to incorporate server holding and set up events by [8].

The following is a list of how this paper is organized. The model's description and discussion as a QBD process are presented in section 2. In part 3, we calculated the queue length's stationary distribution. The stationary queue length and waiting time stochastic decomposition structures are provided in Section 4. Section 5 concludes with numerical demonstrations.

2. Model description

We analyses a queuing model in which consumers arrive at a rate λ according to the Poisson process. During the regular busy period (RBP), the service time is exponentially distributed with a mean of $\frac{1}{\mu_b}$ subject to the condition that $\lambda < \mu_b$. If there are no customers waiting in the system, the server will take WV. Customers are served at an exponential rate μ_v for the WV period, where $\mu_v < \mu_b$ with the parameter θ, the duration of the WV period is also exponentially distributed. The multiple working vacation (MWV) policy mandates the server to keep taking vacations until it finds at least one consumer waiting in the system at the time the vacation ends. If the system is not empty, it changes his service rate from μ_v to μ_b; otherwise, it moves on to the next WV.

When the server is on a WV, the customers are supposed to be impatient, rather than completely stopping service during the WV time frame, the server works at a decreased pace, and consumers feel impatient as a result of the sluggish service rate. Each new consumer sets in motion an independent impatient timer T, which is exponentially distributed at a rate of ξ. If the customer's service is not completed before the timer runs out, the consumer will leave the line and will never return. Furthermore, we anticipated that the failure would occur just during the WV time. Breakdown follows a negative exponential distribution with α as well. When a server fails, it will be sent to be repaired. The fixing procedure is based on Poisson's rate β. The server is now ready for rendering service in RBP after the repair process is completed.

Let Q(t) denote the number of customers in the system at time t, and J(t) denote the server situation, which is defined as follows:

$$J(t) = \begin{cases} 0, & \text{at time t, the server is in WV period,} \\ 1, & \text{at time t, the server is in RBP,} \\ 2, & \text{at time t, the server is in Breakdown} \end{cases}$$

Then a Markov Process with continuous time state space is defined by $\{(Q(t), J(t)); t \geq 0\}$

$$\Omega = \{(k, j): k = 0,1,2,\ldots, j = 0,1\} \cup \{0,2\}$$

The infinitesimal generator is given by using the lexicographical order for the states.

$$Q = \begin{pmatrix} A_{00} & A_{01} & & & \\ B_{10} & A & C & & \\ & B & A & C & \\ & & B & A & C \\ & & \vdots & \vdots & \vdots \end{pmatrix}$$

where

$$A_{00} = \begin{pmatrix} -(\lambda+\alpha) & \alpha \\ 0 & -\beta \end{pmatrix}, \quad A_{01} = \begin{pmatrix} \lambda & 0 \\ 0 & \beta \end{pmatrix}, \quad B_{10} = \begin{pmatrix} \mu_v + \xi & \alpha \\ \mu_b & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} -(\lambda + \mu_v + n\xi + \theta + \alpha) & \theta \\ 0 & -(\lambda + \mu_b) \end{pmatrix}, \quad n \geq 1, E = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix},$$

$$B = \begin{pmatrix} \mu_v + n\xi & 0 \\ 0 & \mu_b \end{pmatrix}, \quad n \geq 2$$

The Markov Chain $\{(Q(t), J(t)); t \geq 0\}$ is a QBD process, according to Q's matrix structure. To understand the QBD process, The smallest non-negative solution of the matrix quadratic equation, which is assumed to represent the rate matrix and denoted by R, must first be found..

$$R^2B + RA + E = 0 \quad (1)$$

The express layout of R is presented in the following lemma.

Lemma 1. The minimal non-negative solution of matrix quadratic equation (1) has the following expression if $\rho = \frac{\lambda}{\mu} < 1$

$$R = \begin{pmatrix} a & \frac{\alpha + r\theta}{\mu_b(1-a)} \\ 0 & \rho \end{pmatrix} \quad (2)$$

where

$$a = \frac{(\lambda + \alpha + \mu_v + n\xi + \theta) - \sqrt{(\lambda + \alpha + \mu_v + n\xi + \theta)^2 - 4(\mu_v + n\xi)\lambda}}{2(\mu_v + n\xi)}; \quad n \geq 1$$

Proof: We can assume that the solution matrix R has the same design as A, B, and C in (1) because they are all upper triangular matrices.

$$R = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

When the values of R^2 and R are substituted into (1), the result is

$$\left.\begin{aligned} &(\mu_v + n\xi)a_{11}^2 - (\lambda + \alpha + \mu_v + n\xi + \theta)a_{11} + \lambda = 0, \ \ n \geq 1 \\ &\mu_b a_{22}^2 - (\lambda + \mu_b)a_{22} + \lambda = 0 \\ &(a_{11} + a_{22})a_{12}\mu_b + \theta a_{11} - (\lambda + \mu_b)a_{12} + \alpha = 0 \end{aligned}\right\} \quad (3)$$

Take $a_{11} = a$ (the other root is invalid because it is bigger than 1) in the first equation to get the smallest non-negative solution of (1). In the second equation of $a_{22} = \rho$ (the other root is invalid because $a_{22} = 1$), we get $a_{22} = \rho$. (3). The fact that $0 < a < 1$ exists is self-evident. We get by substituting a and ρ into (3)'s last equation.

$$a_{12} = \frac{\alpha + a\theta}{\mu_b(1-a)}$$

Because a fulfils the conditions of the following equation

$$(\mu_v + n\xi)a^2 - (\lambda + \theta + \alpha + \mu_v + n\xi)a + \lambda = 0, \qquad n \geq 1$$

When both sides of this equation are divided by a, we get

$$\lambda + \theta + \alpha + (\mu_v + n\xi)(1 - a) = \frac{\lambda}{a}, \;\; n \geq 1$$

Equivalently, we have

$$\frac{\theta+\alpha}{1-a} + \mu_v + n\xi = \frac{\lambda}{a}, \;\; n \geq 1 \quad (4)$$

Theorem 1. If and only if $\rho < 1$, the QBD process $\{(Q(t), J(t)); t \geq 0\}$ is positive recurrent.

Proof : According to neuts (1981), the QBD process $\{(Q(t), J(t)); t \geq 0\}$ is positive recurrent if and only if the rate matrix R's spectral radius SP(R) is less than 1 and the set of equations $(x_0, x_1, x_2)B[R] = 0$ has a beneficial outcome.

$$B[R] = \begin{bmatrix} A_{00} & A_{10} \\ B_{10} & RB + A \end{bmatrix} = \begin{bmatrix} -(\lambda+\alpha) & \alpha & \lambda & 0 \\ 0 & -\beta & 0 & \beta \\ \mu_v + \xi & \alpha & \frac{-\lambda}{a} & \frac{\alpha+\theta}{1-a} \\ \mu_b & 0 & 0 & -\mu_b \end{bmatrix} \quad (5)$$

B[R] is a finite-state irreducible and aperiodic generator. As a result, $(x_0, x_1, x_2)B[R] = 0$ has a beneficial outcome. Thus, if and only if, process $\{(Q(t), J(t)); t \geq 0\}$ is positive recurrent.

$$SP(R) = max(a, \rho) < 1$$

3. Stationary distribution of queue length

Let (Q,J) be the stationary limit of the QBD process if $\rho < 1$. $\{(Q(t), J(t)); t \geq 0\}$

$$\pi_k = (\pi_{k0}, \pi_{k1}), \quad k \geq 1$$

$$\pi_{kj} = P\{Q = k, J = j,\} \; k \geq 0, \;\; J = 0,1$$

Theorem 2. The Stationary Probability Distribution of $\{(Q(t), J(t)); t \geq 0\}$ is $\rho < 1$ if

$$\pi_{k0} = Ka^k, \;\; k \geq 0,$$

$$\pi_{K1} = K\left[\frac{\alpha+a\theta}{\mu_b(1-a)}\sum_{j=0}^{k-1} a^j \rho^{k-1-j} + \frac{\rho^{k-1}}{\mu_b}(\theta+\alpha-(1-a))\right],\ k \geq 1 \qquad (6)$$

where

$$K = \frac{\mu_b(1-a)^2(1-\rho)}{\mu_b(1-a) - a(1-a)(\mu_v+n\xi) + 2\alpha - 3a\alpha + a^2 + \theta(1-a)},\ \ n \geq 1$$

Proof: Using Neuts' (1981) matrix geometric solution method, we get

$$\pi_k = (\pi_{k0}, \pi_{k1}) = (\pi_{10}, \pi_{11})R^{k-1},\ \ k \geq 1 \qquad (7)$$

and $(\pi_{00}, \pi_{20}, \pi_{10}, \pi_{11})$ fulfils the set of equations below

$$(\pi_{00}, \pi_{20}, \pi_{10}, \pi_{11})B[R] = 0$$

By subbing B[R] in (5) into the previous connection, we get

$$-(\lambda+\alpha)\pi_{00} + (\mu_v+\xi)\pi_{10} + \mu_b\pi_{11} = 0$$

$$\alpha\pi_{00} - \beta\pi_{20} + \alpha\pi_{10} = 0$$

$$\lambda\pi_{00} - \frac{\lambda}{a}\pi_{10} = 0$$

$$\beta\pi_{20} + \left(\frac{\alpha+\theta}{1-a}\right)\pi_{10} - \mu_b\pi_{11} = 0$$

Using $\pi_{00} = K$, solve the above equations in terms of $\pi_{00} = K$.

$$\pi_{00} = K, \qquad \pi_{10} = Ka, \qquad \pi_{20} = \frac{\alpha K(1+a)}{\beta} \qquad \pi_{11} = K\left[\frac{\theta+\alpha(2-a^2)}{\mu_b(1-a)}\right]$$

From (2), we obtain

$$R^k = \begin{bmatrix} a^k & \frac{\alpha+a\theta}{\mu_b(1-a)}\sum_{j=0}^{k-1} a^j\rho^{k-1-j} \\ 0 & \rho^k \end{bmatrix},\quad k \geq 1$$

Similarly

$$R^{k-1} = \begin{bmatrix} a^{k-1} & \frac{\alpha + a\theta}{\mu_b(1-a)} \sum_{j=0}^{k-2} a^j \rho^{k-2-j} \\ 0 & \rho^{k-1} \end{bmatrix}, \quad k \geq 1$$

We get (6) by substituting (π_{10}, π_{11}) and the matrix equation R^{k-1} into (7), It's also mentioning that the normalization condition can be used to compute the constant factor K.

4. Stochastic decompositions

We regularly attempt to split the amounts of interest into distinct pieces in order to achieve a better correlation with the previously existing queuing models. The impact of system vacation on system execution lists, such as mean queue length and mean waiting times, is highlighted by stochastic decomposition structures in vacation queuing models. We try to conduct an equivalent decomposition for the system under investigation.

Theorem 3. If $\rho < 1$ and $\mu_b > \mu_v$, $Q = Q_0 + Q_d$where Q_0 is the quantity of consumers in an exemplary M/M/1 queue in steady state and follows a geometric distribution with parameter $(1-\rho)$, and Q_d is the extra number of customers has an adjusted geometric distribution.

$$Q(z) = \frac{(1-\rho)}{(1-\rho z)} K^* \left[(1-a) + (a - \rho + \frac{\alpha + a\theta}{\mu_b(1-a)} + \frac{(\theta + \alpha(1-a))(1-az)}{\mu_b})(\frac{z(1-a)}{(1-az)})\right] \qquad (8)$$

$$where \quad K^* = \frac{K}{(1-a)(1-\rho)} \qquad (9)$$

Proof. From (6), the PGF of Q can be written as follows

$$Q(z) = \sum_{k=0}^{\infty} \pi_{k0} z^k + \sum_{k=1}^{\infty} \pi_{k1} z^k$$

$$= K\left[\frac{1}{1-az} + \frac{(\alpha + a\theta)z}{\mu_b(1-a)(1-\rho z)(1-az)} + \frac{(\theta + \alpha(1-a))z}{(1-\rho z)\mu_b}\right]$$

$$= \frac{(1-\rho)}{(1-\rho z)} K^* \left[\frac{(1-\rho z)}{(1-az)}(1-a) + \frac{(\alpha + a\theta)z}{\mu_b(1-az)(1-a)} + \frac{z(\theta + \alpha(1-a))}{\mu_b}(1-a)\right]$$

$$Q(z) = \frac{1-\rho}{1-\rho z} K^* \left[(1-a) + (a - \rho + \frac{\alpha + a\theta}{\mu_b(1-a)} + \frac{(\theta + \alpha(1-a))(1-az)}{\mu_b})(\frac{z(1-a)}{(1-az)})\right]$$

The above condition demonstrates.

$$Q(z) = K^* \left[(1-a) + (a - \rho + \frac{\alpha + a\theta}{\mu_b(1-a)} + \frac{(\theta + \alpha(1-a))(1-az)}{\mu_b})(\frac{z(1-a)}{(1-az)})\right] \quad n \geq 1$$

is a PGF. $Q_d(z)$ is expanded into a power series of z. The distribution of an additional number of consumers $Q_d(z)$ is obtained. Theorem 3 gives us the stochastic decomposition structure.

$$E(Q_d) = K^*\left[1-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}-\frac{\theta+\alpha(1-a)}{\mu_b}a\right]\frac{1}{1-a}$$

$$E(Q) = \frac{\rho}{1-\rho}+K^*\left[1-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}-\frac{\theta+\alpha(1-a)}{\mu_b}a\right]\frac{1}{1-a}$$

Theorem 4. If $\rho < 1$ and $\mu_b > \mu_v$, $W = W_0 + W_d$, is the sum of two independent variables for an arrival's stationary waiting time W, where W_0 is the waiting time of an arrival in a corresponding classical M/M/1 queue and is exponentially distributed with parameter $\mu_b(1-\rho)$ and W_d is the additional delay with the LST defined by

$$W_d^*(s) = K^*\left[(1-a)+(a-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}+\frac{(\theta+\alpha(1-a))(1-a-sa)}{\lambda\mu_b})\frac{\frac{\lambda}{a}-\lambda}{\frac{\lambda}{a}-\lambda+s}\right] \qquad (10)$$

Proof. The PGF of the quantity of consumers Q can be written as, based on theorem 3.

$$Q(z) = \frac{(1-\rho)}{(1-\rho z)}K^*\left[(1-a)+(a-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}+\frac{(\theta+\alpha(1-a))(1-az)}{\mu_b})(\frac{z(1-a)}{(1-az)})\right] \qquad (11)$$

We get $z = 1-\frac{s}{\lambda}$ in (11) by taking $z = 1-\frac{s}{\lambda}$.

$$\frac{1-\rho}{1-\rho z} = \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+s}, \quad \frac{1-a}{1-az} = \frac{\frac{\lambda}{a}-\lambda}{\frac{\lambda}{a}-\lambda+s}$$

Subbing the above outcome into (11), we get

$$W^*(s) = \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+s}K^*\left[(1-a)+(a-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)} + \frac{(\theta+\alpha(1-a))}{\mu_b}\left(1-a\left(1-\frac{s}{\lambda}\right)\right))\left(1-\frac{s}{\lambda}\right)\frac{\frac{\lambda}{a}-\lambda}{\frac{\lambda}{a}-\lambda+s}\right]$$

$$= \frac{\mu_b(1-\rho)}{\mu_b(1-\rho)+s}K^*\left[(1-a)+(a-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}+\frac{(\theta+\alpha(1-a))(1-a-sa)}{\lambda\mu_b})\frac{\frac{\lambda}{a}-\lambda}{\frac{\lambda}{a}-\lambda+s}\right]$$

Where

$$W_d^*(s) = K^*\left[(1-a)+(a-\rho+\frac{\alpha+a\theta}{\mu_b(1-a)}+\frac{(\theta+\alpha(1-a))(1-a-sa)}{\lambda\mu_b})\frac{\frac{\lambda}{a}-\lambda}{\frac{\lambda}{a}-\lambda+s}\right]$$

As a result, this establishes that $S_d^*(s)$ is an LST.

5. Numerical results

This section shows a graphical depiction of the above-mentioned results and discusses the impact of parameters on the system's performance measurements. $\mu_b = 10$, $\lambda = 3$, $\theta = 0.4$ and $n = 10$ are our assumptions. Figures 1 and 2 show the impact ofμ_v and ξ on $E(Q)$, respectively. Figure 1 shows that for fixed values of ξ, $E(Q)$ decreases as the WV service rate μ_v increases. Figure 2 shows the influence of μ_v and α on $E(Q_d)$, respectively. Figure 2 shows that at fixed levels of α, $E(Q_d)$ decreases as the breakdown rate increases. From the numerical results, the illustration of parameters on the system performance measures is well presented. The results are acceptable to practical situations.

Figure 1: E(Q) versus μ_v

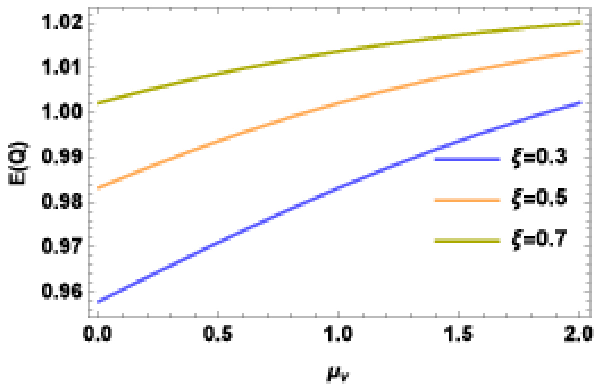
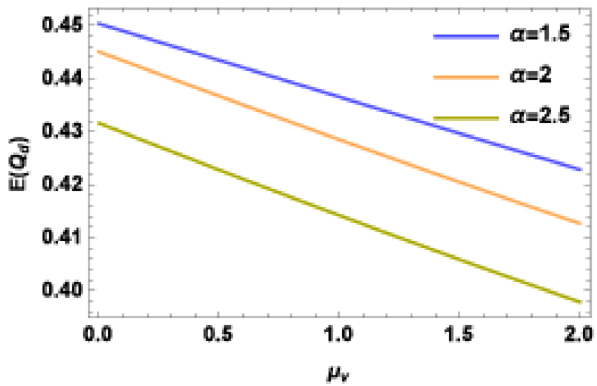


Figure 2: $E(Q_d)$ versus μ_v



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