Original Article

Mathematical Modeling And Analysis Of Stability In Dynamical System Critically Affected By Increasing Rate Of Infertility In Species: Reproductive Toxicology

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Abstract

In this paper, the solution of a nonlinear differential model is obtained to find out the critical solution for a reprotoxic dynamical system. The dynamical system is about a habitat in which some of the species are living under the harmful effect of reprotoxin emitting from the external sources. Here, we considered the simultaneous effects of reprotoxins, one of which is considered as the more reprotoxic than the other or becoming more reprotoxic in the presence of the other. Uptake of reprotoxins by the biological species altered the various functions of the organs. Also it produces its effects on the organs related to the reproductive system which is not only complex but the process of reproduction performed in multiple steps. These reprotoxicity consequences can occur in various forms including additive, combined, or synergistic form, and found that the harmful effect exhibited is more effective than the single one. Here, we obtain the stability analysis at the equilibrium point in which the numerical stability for the proposed model characterizes the stability of a dynamical system in the region of attraction. The mathematical structures stability withstand the decreasing of the steady state of the dynamical system for the changing value of uncontrolled parameters especially the increase rate of the reprotoxin in the environment. This will help in analyzing the energy dissipation capacity of the dynamical model undergoing the effect of controlled and uncontrolled parameters, thus allowing stopping the damage in the structural parameters responsible for divergent of the trajectory path away from the equilibrium point. Another approach that is being utilized in the dynamical system is to provide the external visualization of variation of varying parameters in multidirectional path i.e. changing one and keeping the other parameters constant, estimating the point where the diminishing or the collapse of the system can occur. The main purpose of this paper is to examine the stability both local as well as global of the dynamical system via using mathematical tools and mathematical structural stability to understand the movement of trajectory path towards the fixed point in the attractor basin of the dynamical system.

Keywords: Mathematical model, Reproductive toxicant, Reproduction, Biological species, 3D visualization

1. INTRODUCTION

Industrialization, intellectual disordered, globalization, modernization and devastation is severely impacting the atmosphere. Utilization of fertilizer, pesticides and garbage yield by the different activities are producing the adverse impact on the reproductivity of various species. The variety of toxicants brings the physiological impacts, instant impacts on reproduction system and adverse effects on embryos. The most common adverse impact on embryos contain failure at different steps of the reproductive process, mortality rate, teratological effect producing skeletal abnormalities and hormonal imitating of estrogens yields defects in nervous system through various mechanism. Lethal and sub lethal stress, infertility mortality, repression of egg configuration etc. are covered in the range of the chemical effect on biological species. The varieties of reprotoxin yield adverse effects in reproduction which includes man-made contaminations, organophosphate pesticides, organochlorine pesticides etc. Polychlorinated biphenyls (PCBs), o, p'-DDT a combination of organochlorines contaminations have been considered as environmental estrogen affecting biological population. Many of the researchers have performed their research on one or two toxins on the species including the simultaneous impact of two toxins by the utilization of the nonlinear differential model [5, 26, 28, & 36]. The developmental growth rate of the species is linearly dependent on the uptake quantity or doses of the reprotoxin by the subclass in species. Most of the research works have shown some lackness which was later on corrected like the carrying capacity of species in environment for increasing rate of reprotoxins. Hallam et.al (1983a) investigated is based on first order toxicant absorption kinetics, the logistic population model, and a linear dose-response function. It is designed to begin where standard destiny models end, namely with a toxicant [21] concentration in the environment.

Population biomass (or density), concentration in an organism, and concentration in the environment are the three state variables. Beyond, Shukla and Dubey (1996) have suggested a model and shown that the impact of two toxins are producing [36] more toxic effect on the species as compared to the single toxicant in which one is more toxic than the other. Freedman & Shukla has developed [16] equation with relation to the mass of the environment, and it includes both population density and toxicant concentration in the population for the toxicant concentration in the environment. Because each of the equations involves all of these variables, researchers believe that identifying all concentrations makes the model of the ecotoxicological problem more visible. Dickman & Rygiel investigate that sediments collected near the source of an oil [14] leak were found to be very hazardous. The effects of these oil and heavy metal contaminated sediments on a midge (Chironomid) larvae population were studied. Agrawal and Shukla (2012) observed that as the toxicity of toxicants from external sources increased, so did their toxicity in the [4] atmosphere, resulting in the dysfunction of various parts of biological species. Ronit and Eldad (1998) investigated the effects of inhibiting [34] acetyl-cholinesterase and its activities on reproductive morphological and behavioral changes. Carbon dioxide is being removed from the atmosphere, but worldwide forest [29] biomass is falling at an alarming rate due to human activity. Sensitivity analysis shows that increasing the rate of deforestation increases the atmospheric concentration of CO2. Guillette has suggested the abnormalities in alligator and the failure in [19] reproduction has been changed analogous that the molecular or the cellular positions of this process are modified. So the changes observed are not reversible but permanent. It has been noted that majority of the research contains the effect of the single toxicant however under the actual condition the species are under the effects of more than single toxicant on the biological species. The species is consistently acted upon by various toxicants with the harmful effect including the study of the growth of Eurytemora affinis. Barlow and Sullivan (1981) found that chemicals such as lead, methyl mercury, beryllium, carbon monoxide, benzene, chlordecone, dibromochloropropane, and hexa-chlorobenzene [7] can harm both men and women. Reproductive problems caused by industrial chemical exposure do not only affect pregnant women, but can affect both men and women at any time. Hallam and Clark (1981) found that either a decreasing growth rate or a deteriorating environment ensures population extinction. A large average growth rate might result in [20] extinction, whereas a small average growth rate can result in persistence. Colodey & Wells (1992) found that PCBs, dibenzofurans and dibenzodioxins are the toxicant emitted from the external sources through industrial waste disposition, waste incineration, or as pollutants. PCB is a historical problem and largely continued to expose into the atmosphere. Kraft-process paper and incinerators pulp bleaching mills are sources of dibenzofurans and dibenzodioxins, but the paper mill helps to reduce the dioxins to nonchlorine bleaching processes. Okamura and Aoyama (1994), researched the combined [31] effects of two metals cadmium and chromium and found the consequence of the combined effect of the toxicant. Hallam et.al (1983b) examines that the ecological impacts of a harmful [22] contaminant on a population that has been exposed to it. Analytic techniques used to extract model conclusions point to processes of persistence and extinction. The findings back up the theory that the toxicant-population-dose-response system can result in several stable equilibriums. Dickman and Rygiel (1996) studied the effects of toxic metals and oily debris in the Niagara River on the larvae of an invertebrate midge species (chironomids) [15], found that around 26% of the chironomids were deformed from 10 to 800 meters downstream. Despite the fact that the species is impacted by several toxicants, the majority of investigations, including the study of Eurytemora affinis development, have focused on a single toxicant. Veeramachaneni (2001) discovered that a number of [38] chemical substances contaminate the male reproduction process. They have endocrine disrupting effects (EDCs). EDCs have the potential to disrupt the development of the reproductive system. Both sexes are affected, and the development of their reproductive tracts is altered. Laboratory knowledge gathered over past twenty years indicates that reproduction of birds is also affected by the mechanisms of the toxic action of the organochlorine. The fields surveying the harmful effects seen in birds are veritably used in foretelling harmful effects in other wildlife, snarling turtles, or alligators. Cairns (1990) research on the [9] combined effect of ammonium and chlorine on submarine individually and in combination. Also it has been found that 20% of the class richness of the protozoans is decreases with the compounding toxicant concentration. Atlas (1991) investigated the biological population density to a toxicants and constitute that the taxonomic and heritable [2] kinds of this population were lots lower than the peaceful authority neighborhoods. The ecological impacts of a harmful contaminant on a population that has been exposed to it. Analytic techniques used to extract model conclusions point to processes of persistence and extinction. The findings back up the theory that the toxicant-population-dose-response system can result in several stable equilibriums. Widdows collected mussel samples from various parts of the Venice Lagoon and determined that they were contaminated by various toxicants [39] (for example, chromium, mercury, cadmium, iron, manganese, or chlorinated hydrocarbons), resulting in a decrease in the population of species compared to other parts of the Lagoon. Abdul Rahman and Habib (1989) investigated the Allelopathic harmful effects of alfalfa Medicago [1] Sativa on blady grasses (Imperata cylindrical) and found that the toxicant produced is not in very large quantity but is strong to influence on the soil microflora producing a harmful effect on the growth of the source as well as on neighboring plants. Huaping and Ma Zhien (1991) investigated the effect of pollutants in [24] two biological competitive environments using mathematical modeling. Dubey also studied that in chemical defense mechanism two species complete to occupies space and releases toxicant such that toxicant produce by one is affecting the life of other and vice versa. Similarly DBCP, EDB both are volatile and bio-accumulated [12] and are observed via the dermal and oral routes. In human, EDB decreases the reproductive capability and lower down the percentage of the number of species, and increasing certain types of deformities inside the reproductive part of the biological species. Chattopadhyay (1996) assumed and discussed a nonlinear model in a dynamic system in which the research was related to the simultaneous effect of two [10] toxicants and one toxicant were considered to producing more harmful effects than the other toxicant on a biological population. DDT is not only responsible the various [17] diseases in species but this pollutant also produces adverse effect in the reproduction birds. Other persisting organochlorine diseases with substantiated effects contained dieldrin, mirex, toxaphene, hexa-chlorobenzene, endrin, chlordane, and lindane. Patil and David (2010) also research on the study of fish in created [32] toxic media and found that they move through various angles like circular movements, haphazard, hyperexcitability and mostly the movements observe at the bottom losing their equilibrium. Hartwell (1993) research the growth rate of Eurytemora affinis (copepoda) confirms the high fecundity when following through the different [23] chambers at various contaminated localities in Chesapeake Bay which indicated the qualities of river at suitable place and between the locations. Sun has (2009) exposed the outer and external morphological malformation that are the indicators as [37] biomarkers in fish from various contaminant rivers or channels in Taiwan. Lindley (1999) have suggested [28] that when two organochlorine compounds 1, -2dichlorobenzene (DCB) a non-polar tranquilizing and pentachlorophenol (PCP), a respiratory uncouple were selected for toxicants eggs of estuarine as well as neritic planktonic calanoid copepods. It is concluded that as the concentration of the two compound increases it reduces the percentage of the eggs after exposed to pesticide. Woin and Bronmark (1992) suggested that different proportion of DDT and MCPA produces various effects on the reproduction of the [40] snails when collected from a eutrophic pond located in Southern Sweden and have seen that there is propound effect on the distribution of snails however have no effects on the mortality rate of the species (snails). Kumar (2016a) investigated a model to examine the [6] effect of toxicants on the subclass of the species. He suggested that the effect of a single toxicants producing the harmful effect on the reproduction of the subclass in species emitted by species itself (e.g., vehicular exhaust, radioactive wastes, industrial waste, and familial waste, compost, etc.,)and external sources (e.g., heavy metal, volcanic eruptions, forest fires, etc.,). In this investigation it has been found that sometime the biological species don’t have any morphological deformity but are affected internally such as decline in the reproductive capability, asthma, cancer etc. resulting in the increasing rate of mortality of the biological species. Deluna & Hallam establish and test three models of population-toxicant [13] interactions, each of which includes an explicit representation of a dynamic resource. These models' asymptotic behaviour are investigated, persistence and extinction criteria are developed, and common characteristics of the behaviour are compared. Kumar (2016) have found that the impact of combined effect of two toxicant on the [26] biological species. It was observed that the density of the total population decreases with the increasing level of toxicant in the atmosphere. In this investigation; it is also assumed that the reproductive toxicant is supposed to be discharged into the atmosphere by the biological species itself. Agrawal (2020) investigated the impact of reprotoxin on the biological species discharge into the environment from the sources and found that as the system becomes more reprotoxic, the [3] infertility in the biological population increases due to the increases in the deformities in the reproduction process and this decreases the total population below the carrying capability. Singh (2021) investigated that increase in the reproduction process rises the population rate of various [35] species and also observed that with the rise in the reprotoxin into the environment, the reproduction rate decreases in species and gets lethal and sub-lethal effects because of this the population decreases below the carrying capacity. Now, we assume and analyze a mathematical model to investigate the simultaneous effect of two reproductive toxicants discharged into the environment from external sources in a subclass of biological species. We have expected that the biological species are antagonistically influenced by this reproductive process and a few species lose their capability of reproduction, decreases the total population density, decreasing the number of inhabitants in numerous species. Reproductive failure is a worldwide emerging problem that threatened most of the species by putting them in an endangered zone and is also responsible for balancing the ecosystem. This proposed model confirms some of the previous research and discusses to develop the further scope of the dynamic model concluded in the various plots of simulation. The purpose of this paper is to discuss the simultaneous effect of two reprotoxin from the external sources, including mathematical structural stability of dynamical population and thus develop more understanding regarding the movement of trajectory path near the equilibrium points in the region of attraction or attractor basin.

2. MATHEMATICAL MODEL

Let us consider a habitat in which the population exists and grows with time (t). Reprotoxin is discharging into the environment in constant proportion and produces various harmful effects on the fertility rate of species. Each of the toxicants discharges into the atmosphere continuously and it decreases due to some washout factors. The uptake concentration is different in different species and harmful effects produced due to it, also vary depending upon their body immunity. It is the observer as the absorption of the reprotoxin in species raises the infertility rate increases and their harmful effect divides the total population into two subclasses, one subclass capable of performing the reproduction process successfully whereas the other subclass loses the reproduction capability due to an increase in the deformities in various parts of reproduction in subspecies. The subspecies of the biological population capable of reproduction is $N_f(t)$ and the subspecies of the biological population losing its capability of reproduction is $N_d(t)$. Here, one reprotoxin $T_1(t)$ discharged in the atmosphere with emission rate Q_1 by biological population itself and the other $T_2(t)$ discharged with emission rate Q_2 at time t. $U_1(t)$& $U_2(t)$ is the uptake rate of concentration of reprotoxin of $T_1(t)$ and $T_2(t)$ respectively by the biological population $N(t)$ at time t. With variable intrinsic growth rate $R(U_1, U_2) = r - r_1U_1 - r_2U_2 > 0$, (where r_1 & r_2 are the decreasing rates of the natural growth rate r ($i.e. r =$ birth rate, $b -$ death rate, d) associated with the uptake of reprotoxin $U_1(t)$, $U_2(t)$ and carrying capability density $K(T_1, T_2)$ associated with atmospheric concentrations $T_1(t)$, $T_2(t)$ of biological population which are formulated and given below:-

$$\frac{dN_f}{dt} = \left[r - (r_1U_1 + r_2U_2) - \frac{1}{K(T_1, T_2)}[rN - (b + \alpha)N_d]\right]N_f$$

$$\frac{dN_d}{dt} = \left(\frac{(r_1U_1 + r_2U_2)N_f}{N_d} - d - \alpha - \frac{1}{K(T_1,T_2)}[rN - (b+\alpha)N_d]\right)N_d$$
$$\frac{dT_1}{dt} = Q_1 - \delta_1T_1 - \gamma_1NT_1 + \pi_1v_1NU_1$$
$$\frac{dT_2}{dt} = Q_2 - \delta_2T_2 - \gamma_2NT_2 + \pi_2v_2NU_2 \quad (1.1)$$
$$\frac{dU_1}{dt} = \gamma_1NT_1 - \beta_1U_1 - v_1NU_1$$
$$\frac{dU_2}{dt} = \gamma_2NT_2 - \beta_2U_2 - v_2NU_2$$
$$N(t) = N_f(t) + N_d(t),\ N_f(0), N_d(0),\ \ T_1, T_2 \geq 0,\ \ U_1 \geq c_1T_{1,}\ U_2 \geq c_2T_{2,}, 0 < \pi_1 < 1, 0 < \pi_2 < 1$$

Here, α is the death rate of deformed population density as a result of simultaneous effect of two reproductive toxicant discharged by both the external sources. δ_1 & δ_2 are the natural depleting rates coefficient of $T_1(t)$ & $T_2(t)$ respectively, β_1 & β_2 denote the natural depleting coefficient of $U_1(t)$ & $U_2(t)$ respectively, γ_1 & γ_2 are the depletion rates coefficient of $U_1(t)$ and $U_2(t)$ respectively due to the uptake of reproductive toxicant by the total population N, v_1 & v_2 are the depleting coefficient of $U_1(t)$ & $U_2(t)$ respectively due to decompose or decay of some population of N and π_1 & π_2 are fractions of the depletion of $U_1(t)$ & $U_2(t)$ due to decay of some members of N which reintegrated with atmospheric toxicant concentrations $T_1(t)$ & $T_2(t)$. $K(T_1, T_2)$, a decreasing function of N to obtain the carrying capability of the environment.
Since, $N(t) = N_f(t) + N_d(t)$, then the model eq. (1.1) reduced as:

$$\frac{dN}{dt} = [rN - (\alpha + b)N_d]\left[1 - \frac{N}{K(T_1,T_2)}\right]$$

$$\frac{dN_d}{dt} = (r_1U_1 + r_2U_2)(N - N_d) - (\alpha + d)N_d - [rN - (\alpha + b)N_d]\frac{N_d}{K(T_1,T_2)}$$
$$\frac{dT_1}{dt} = Q_1 - \delta_1T_1 - \gamma_1NT_1 + \pi_1v_1NU_1$$
$$\frac{dT_2}{dt} = Q_2 - \delta_2T_2 - \gamma_2NT_2 + \pi_2v_2NU_2 \quad (1.2)$$
$$\frac{dU_1}{dt} = \gamma_1NT_1 - \beta_1U_1 - v_1NU_1$$
$$\frac{dU_2}{dt} = \gamma_2NT_2 - \beta_2U_2 - v_2NU_2$$

$$N(t) = N_f(t) + N_d(t),\ N_f(0), N_d(0),\ \ T_1, T_2 \geq 0,\ \ U_1 \geq c_1T_{1,}\ U_2 \geq c_2T_{2,}, 0 < \pi_1 < 1, 0 < \pi_2 < 1$$

In the proposed model eq. (1.1), $c_1, c_2 > 0$ are constants for the initial concentration $U_1(0)$ & $U_2(0)$ with the initial density of biological population $N(0)$. The carrying capability function $K(T_1, T_2)$ has the following properties:

$$K(0,0) = K_0 > 0, \qquad K(T_1,T_2) > 0, \qquad \frac{\partial K(T_1,T_2)}{\partial T_1} < 0, \qquad \frac{\partial K(T_1,T_2)}{\partial T_2} < 0$$

If both T_1& T_2 increase into the different habitats from the external sources then the population density decreases and it goes below the carrying capacity. The sequence is form in which the harmful effects of the two toxicants can be easily seen and is discussed below:-

$$K(T_1,T_2) < \frac{\partial K(T_1,T_2)}{\partial T_2} < \frac{\partial K(T_1,T_2)}{\partial T_1} < K(0,0) \qquad (1.3)$$

Here, K_0 is the highest carrying capacity of biological species for proposed system eq. (1.2) when the reprotoxic natures of the atmosphere are almost negligible. That is the biological species are living environment free from reprotoxic effect. Now we investigate steady rate of the dynamical model eq. (1.2) showing the trajectory movement of the model towards the fixed points, and to obtain the movement we determine the region of attraction Ω. The region of attraction for the proposed system eq. (1.2), clearly shown in lemma eq. (1.1)
Lemma.2.1.The region of attraction of proposed system is denoted by Ω and

$$\Omega = \left\{(N, N_{d,}T_1, T_2, U_1, U_2): 0 \leq N \leq K_0, 0 \leq N_d \leq \frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]}, 0 \leq (T_1 + T_2 + U_1 + U_2) \leq \frac{(Q_1 + Q_2)}{\delta_m}\right\}$$

where $\delta_m = \min(\delta_1, \delta_2, \beta_1, \beta_2)$ is the attractor basin which attracts all solutions initiating in the interior of the positive orthant.

Proof: The first equation of model eq. (1.2):

$$\frac{dN}{dt} = [rN - (\alpha + b)N_d]\left[1 - \frac{N}{K(T_1,T_2)}\right]$$

Gives:

$$\frac{dN}{dt} \le rN\left\{1-\frac{N}{K(T_1,T_2)}\right\}$$

Thus, $\limsup_{t\to\infty} N(t) \le K_0$.

The second equation of model eq. (1.2):

$$\frac{dN_d}{dt} = (r_1U_1 + r_2U_2)(N - N_d) - (\alpha + d)N_d - [rN - (\alpha + b)N_d]\frac{N_d}{K(T_1,T_2)}$$

Gives:

$$\frac{dN_d}{dt} \le \quad \le r_1\,NU - r_1N_dU - (\alpha + d)N_d - [rN - (\alpha + b)N_d]\,\frac{N_d}{K(T)}$$

$$\le r_1NU - r_1N_dU - [rN - (\alpha + b)N_d]\,\frac{N_d}{K(T)}$$

$$\le r_1NU - r_1N_dU + (\alpha + b)\,\frac{N_d^2}{K(T)} \quad \text{Thus}, \limsup_{t\to\infty} N_d(t) \le \frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]}$$

.

The sum of the last four equations of the dynamic model eq. (1.2), gives:

$$\frac{dT_1}{dt} + \frac{dT_2}{dt} + \frac{dU_1}{dt} + \frac{dU_2}{dt} = (Q_1 + Q_2) - \delta_1T_1 - \delta_2T_2 - \beta_1U_1 - \beta_2U_2 - (1 - \pi_1)\nu_1NU_1 - (1 - \pi_2)\nu_2NU_2$$

$$\le (Q_1 + Q_2) - \delta_m(T_1 + T_2 + U_1 + U_2).$$

$$\text{Where,} \quad \delta_m = \min\{\delta_1, \delta_2, \beta_1, \beta_2\}, \text{Thus}, \limsup_{t\to\infty}(T_1 + T_2 + U_1 + U_2) \le \frac{(Q_1 + Q_2)}{\delta_m}.$$

3. STABILITY ANALYSIS

3.1. Equilibrium points of model

Now we determine the equilibrium points or the stability for the proposed model in the region of attraction and then characterized each of the fixed points as stable based on the trajectory behavior of the line of path moving in the attractor region. In the neighborhood of each fixed point, the dynamical behavior of the system undergoes various changes some moment of trajectory path towards the equilibriums point or away from it making the system stable or unstable depending upon the changes in the variables or parameter. To get the best approximate solution of the proposed model of biological species, the eq. (1.2) is solved by analytical means. Hence, this paper contains various analytical methods for the best approximate result of the proposed model. The solution of the system is stable if the output is to control the trajectory movement asymptotically towards the equilibrium points to keep the dynamical model stable. The proposed model has the existence of three different non-negative equilibrium points which are as given below:-

$$E_1 = \left\{0,0,\frac{Q_1}{\delta_1},\frac{Q_2}{\delta_2},0,0\right\}, E_2 = \{\tilde{N},\tilde{N}_d,\tilde{T}_1,\tilde{T}_2,\tilde{U}_1,\tilde{U}_2\}, E_3 = \{N^*,N_d^*,T_1^*,T_2^*,U_1^*,U_2^*\}$$

Now we make an analysis of the proposed system for further analysis. To investigate the stability in the dynamical population of the proposed model, we check the existence and uniqueness of a system of the nonlinear differential equations at equilibrium points $E_1 = \left\{0,0,\frac{Q_1}{\delta_1},\frac{Q_2}{\delta_2},0,0\right\}, E_2 = \{\tilde{N},\tilde{N}_d,\tilde{T}_1,\tilde{T}_2,\tilde{U}_1,\tilde{U}_2\}$ & $E_3 = \{N^*,N_d^*,T_1^*,T_2^*,U_1^*,U_2^*\}$**.** The existence of E_1 is obivious. Now we check the uniqueness and existence at the equilibrium points E_2 & E_3 which would be provide the positive value of $\{\tilde{N},\tilde{N}_d,\tilde{T}_1,\tilde{T}_2,\tilde{U}_1,\tilde{U}_2\}$ & $\{N^*,N_d^*,T_1^*,T_2^*,U_1^*,U_2^*\}$ respectively.

Existence and uniqueness of E_2: Here $\tilde{N},\tilde{N}_d,\tilde{T}_1,\tilde{T}_2,\tilde{U}_1$ & $\tilde{U}_2$ are the positive values moving each of the trajectory paths in attractor basins giving solutions of the system of equations.

$$N = K(T_1,T_2) \qquad (1.4a)$$

$$N_d = K(T_1,T_2) \qquad (1.4b)$$

$$T_1 = \frac{Q_1(\beta_1 + v_1N)}{f_1(N)} = g_1(N) \quad (1.4c)$$

$$T_2 = \frac{Q_2(\beta_2 + v_2N)}{f_2(N)} = g_2(N) \quad (1.4d)$$

$$U_1 = \frac{Q_2\gamma_1N}{f_1(N)} = h_1(N) \qquad (1.4e)$$

$$U_2 = \frac{Q\gamma_2N}{f_2(N)} = h_2(N) \qquad (1.4f)$$

$$\text{where,} \quad f_1(N) = \{\delta_1\beta_1 + (\gamma_1\beta_1 + \delta_1v_1) + \gamma_1v_1(1 - \pi_1)N^2 \qquad (1.4g)$$

$$f_2(N) = \{\delta_2\beta_2 + (\gamma_2\beta_2 + \delta_2v_2) + \gamma_2v_2(1 - \pi_2)N^2 \qquad (1.4h)$$

From eq. (1.4a)

$$F_1(N) = N - K(T_1,T_2) = N - K\big(g_1(N), g_2(N)\big) \quad (1.5)$$

From the above it is clear that

$$F_1(0) < 0 \text{ and } F_1(K_0) > 0 \qquad (1.6)$$

$$F_1(N) = N - K(T_1, T_2) = N - K\big(g_1(N), g_2(N)\big) \quad (1.7)$$
$$\frac{d}{dN}F_1(N) = \left[1 - \left\{\frac{\partial K}{\partial T_1}\frac{dg_1}{dN} + \frac{\partial K}{\partial T_2}\frac{dg_2}{dN}\right\}\right] \quad (1.8)$$
From, equations (1.4c & 1.4d), we get
$$\frac{dg_1}{dN} = \frac{-\gamma_1 Q_1}{f_1^2(N)}\{\delta_1(\beta_1 + v_1 N)^2 + \gamma_1 v_1 \pi_1 \beta_1 N^2\} > 0 \quad (1.9a)$$
$$\frac{dg_2}{dN} = \frac{-\gamma_2 Q_2}{f_2^2(N)}\{(1 - \pi_2)(\beta_2 + v_2 N)^2 + \pi_2 \beta_2^2\} < 0 \quad (1.9b)$$
Equations $(1.8,\ 1.9a\ \&1.9b)$ implies that $\frac{d}{dN}F_1(N) > 0$, as well as root N in system of equations (1.4a – 1.4f) will be unique, if following condition hold, i.e.
$$\frac{\partial K}{\partial T_2}\frac{dg_2}{dN} < 1 + \left|\frac{\partial K}{\partial T_1}\frac{dg_1}{dN}\right| \quad (1.10)$$
From the above equations (1.6, 1.8) it is clear that $F_1(N) = 0$ has a unique root N in the interval $[0, K_0]$ under a certain condition (1.10). Since all the above conditions are satisfied so this suggests that knowing the value of $\widetilde{N}$ all the other values i.e. $\widetilde{N}_d, \widetilde{T}_1, \widetilde{T}_{2,} \widetilde{U}_1$ & $\widetilde{U}_2$ can be computed from the equations $(1.4b - 1.4h)$. From the above it is clear that there exists the value for $\widetilde{N}, \widetilde{N}_d, \widetilde{T}_1, \widetilde{T}_{2,} \widetilde{U}_1$ & $\widetilde{U}_2$ and is positive if the two toxicants are emitted into the atmosphere. Now, we further check for the positive solution at the equilibrium point E_3.

Existence and uniqueness of $\boldsymbol{E_3}$: Here $N^*, N_d^*, T_1^*, T_2^*, U_1^*$ & U_2^* are the positive solutions of the system of equations.
$$N = K(T_1, T_2) \quad (1.11a)$$
$$N_d = \frac{(r_1 U_1 + r_2 U_2)}{(\alpha + b)} K(T_1, T_2) \quad (1.11b)$$
T_1, T_2, U_1& U_2 are same as eqs. $(1.4c - 1.4f)$. Since the value of N in eq. (1.11a) is same as in eq. (1.4.a). So, the existence and uniqueness at equilibrium point $E_3(N^*, N_d^*, T_1^*, T_2^*, U_1^*$ & $U_2^*)$ is same which confirms the existence and uniqueness at the fixed point $E_2\big(\widetilde{N}, \widetilde{N}_d, \widetilde{T}_1, \widetilde{T}_{2,} \widetilde{U}_1, \widetilde{U}_2\big)$.

3.2 Local stability at the equilibrium points

The jacobian matrix M_1 corresponding to the equilibrium point $E_1 = \left(0, 0, \frac{Q_1}{\delta_1}, \frac{Q_2}{\delta_2}, 0,0\right)$ is as given below: -
$$\mathrm{M}_1 = \begin{bmatrix} r & -(\alpha + b) & 0 & 0 & 0 & 0 \\ 0 & -(\alpha + d) & 0 & 0 & 0 & 0 \\ \frac{-\gamma_1 Q_1}{\delta_1} & 0 & -\delta_1 & 0 & 0 & 0 \\ \frac{-\gamma_2 Q_2}{\delta_2} & 0 & 0 & -\delta_2 & 0 & 0 \\ \frac{\gamma_1 Q_1}{\delta_1} & 0 & 0 & 0 & -\beta_1 & 0 \\ \frac{\gamma_2 Q_2}{\delta_2} & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}$$

Here, the eigenvalues of jacobian matrix M_1 are $\mathrm{r}, -(\alpha + d), -\delta_1, -\delta_2, -\beta_1, -\beta_2$. As it is clearly seen that all the eigenvalues are not negative at this equilibrium point, therefore the dynamical system has a saddle point, locally stable & unstable manifolds. The system has unstable manifolds in the direction of N and stable manifold in the direction of $N_d - T_1 - T_2 - U_1 - U_2$ space. Therefore the proposed system is unstable at E_2 fixed point.
The jacobian matrix M_2 at the fixed point $\boldsymbol{E_2} = \widetilde{N}, \widetilde{N}_d, \widetilde{T}_1, \widetilde{T}_{2,} \widetilde{U}_1$ & $\widetilde{U}_2$ is given below:
$$M_2 = \begin{bmatrix} l_{11} & 0 & l_{13} & l_{14} & 0 & 0 \\ l_{21} & l_{22} & l_{23} & l_{24} & 0 & 0 \\ l_{31} & 0 & l_{33} & 0 & l_{35} & 0 \\ l_{41} & 0 & 0 & l_{44} & 0 & l_{46} \\ l_{51} & 0 & l_{53} & 0 & l_{55} & 0 \\ l_{61} & 0 & 0 & l_{64} & 0 & l_{66} \end{bmatrix}$$
$$l_{11} = d + \alpha,\ l_{13} = -(\alpha + d)\left(\frac{\partial K}{\partial T_1}\right)_{E_2},\ l_{14} = -(\alpha + d)\left(\frac{\partial K}{\partial T_2}\right)_{E_2}, l_{21} = (r_1\widetilde{U_1} + r_2\widetilde{U_2}) - r,$$
$$l_{22} = -(r_1\widetilde{U_1} + r_2\widetilde{U_2}) + \alpha + b, l_{23} = -(\alpha + d)\left(\frac{\partial K}{\partial T_1}\right)_{E_2}, l_{24} = -(\alpha + d)\left(\frac{\partial K}{\partial T_2}\right)_{E_2}$$
$$l_{31} = -\gamma_1\widetilde{T_1} + \pi_1 v_1\widetilde{U_1}, l_{33} = -\left(\delta_1 + \gamma_1 K\big(\widetilde{T_1}, \widetilde{T_2}\big)\right), l_{35} = \pi_1 v_1 K\big(\widetilde{T_1}, \widetilde{T_2}\big)$$
$$l_{41} = -\gamma_2\widetilde{T_2} + \pi_2 v_2\widetilde{U_2}, l_{44} = -\left(\delta_2 + \gamma_2 K\big(\widetilde{T_1}, \widetilde{T_2}\big)\right), l_{46} = \pi_2 v_2 K\big(\widetilde{T_1}, \widetilde{T_2}\big)$$
$$l_{51} = \gamma_1\widetilde{T_1} - v_1\widetilde{U_1}, l_{53} = \gamma_1 K\big(\widetilde{T_1}, \widetilde{T_2}\big), l_{55} = -\left(\beta_1 + v_1 K\big(\widetilde{T_1}, \widetilde{T_2}\big)\right),$$

$$l_{61} = \gamma_2 \widetilde{T_2} - v_2 \widetilde{U_2}, l_{64} = \gamma_2 K(\widetilde{T_1}, \widetilde{T_2}), l_{66} = -\left(\beta_2 + \nu_2 K(\widetilde{T_1}, \widetilde{T_2})\right)$$

On solving the above jacobian matrix M_2 we conclude that one of the eigenvalue is non-negative and equal to $2(\alpha + b) > 0$. This shows that not all eigenvalues are negative i.e. one of the eigenvalue is positive and therefore the proposed system of biological species is again unstable at the equilibrium E_2. Now we determine the eigenvalue of jacobian matrix M_3 corresponding to the equilibrium point $E_3 = (N^*, N_d^*, T_1^*, T_2^*, U_1^*, U_2^*)$ is:

$$M_3 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & 0 & m_{33} & 0 & m_{35} & 0 \\ m_{41} & 0 & 0 & m_{44} & 0 & m_{46} \\ m_{51} & 0 & m_{53} & 0 & m_{55} & 0 \\ m_{61} & 0 & 0 & m_{64} & 0 & m_{66} \end{bmatrix}$$

Moreover,

$$m_{11} = -r\left\{\frac{2N^*}{K(T_1^*, T_2^*)} - 1\right\} + \frac{(\alpha + b)N_d^*}{K(T_1^*, T_2^*)}, m_{12} = -(\alpha + b)\left\{1 - \frac{N^*}{K(T_1^*, T_2^*)}\right\},$$

$$m_{13} = -[rN^* - (\alpha + b)N_d^*]\frac{N^*}{K^2(T_1^*, T_2^*)}\left(\frac{\partial K}{\partial T_1}\right)_{E_3}, m_{14} = -[rN^* - (\alpha + b)N_d^*]\frac{N^*}{K^2(T_1^*, T_2^*)}\left(\frac{\partial K}{\partial T_2}\right)_{E_3}$$

$$m_{21} = (r_1 U_1^* + r_2 U_2^*) - \frac{rN_d^*}{K(T_1^*, T_2^*)}, m_{22} = -(r_1 U_1^* + r_2 U_2^* + \alpha + d) + (\alpha + b)\frac{2N_d^*}{K(T_1^*, T_2^*)},$$

$$m_{23} = [rN^* - (\alpha + b)N_d^*]\left[\frac{N_d^*}{K^2(T_1^*, T_2^*)}\right]\left(\frac{\partial K}{\partial T_1}\right)_{E_3}, m_{24} = [rN^* - (\alpha + b)N_d^*]\left[\frac{N_d^*}{K^2(T_1^*, T_2^*)}\right]\left(\frac{\partial K}{\partial T_2}\right)_{E_3},$$

$m_{25} = r_1(N^* - N_d^*), m_{26} = r_2(N^* - N_d^*), m_{31} = -\gamma_1 T_1^* + \pi_1 v_1 U_1^*,\ m_{33} = -(\delta_1 + \gamma_1 N^*),$
$m_{35} = \pi_1 v_1 N^*, m_{41} = -\gamma_2 T_2^* + \pi_2 v_2 U_2^*, m_{44} = -(\delta_2 + \gamma_2 N^*),\ m_{46} = \pi_2 v_2 N^*, m_{51} = \gamma_1 T_1^* - v_1 U_1^*, m_{53} = \gamma_1 N^*,$
$m_{55} = -(\beta_1 + v_1 N^*), m_{61} = \gamma_2 T_2^* - v_2 U_2^*, m_{64} = \gamma_2 N^*, m_{66} = -(\beta_2 + v_2 N^*)$

Here, M_3 is the jacobian matrix corresponding to the fixed point E_3. Therefore, the characteristic equation of M_3 can be written as,

$$P(x) = x^6 + B_1 x^5 + B_2 x^4 + B_3 x^3 + B_4 x^2 + B_5 x + B_6. \quad (1.12)$$

Where,

$$B_1 = -(m_{11} + m_{22} + m_{33} + m_{44} + m_{55} + m_{66})$$

$$B_2 = (m_{11} + m_{22})(m_{55} + m_{66}) + m_{22}(m_{33} + m_{44}) + (m_{33} + m_{55})(m_{44} + m_{66}) + \begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix} + \begin{vmatrix} m_{11} m_{13} \\ m_{31} m_{33} \end{vmatrix} + \begin{vmatrix} m_{11} m_{14} \\ m_{41} m_{44} \end{vmatrix} + \begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix} + \begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}$$

$$B_3 = m_{14} m_{41} m_{33} - m_{11} m_{55} m_{66} - m_{12}(m_{25} m_{51} + m_{26} m_{61}) - m_{22}(m_{33} + m_{55})(m_{44} + m_{66}) + m_{13}\begin{vmatrix} m_{31} m_{35} \\ m_{51} m_{55} \end{vmatrix} + m_{14}\begin{vmatrix} m_{41} m_{46} \\ m_{61} m_{66} \end{vmatrix} - m_{31}\begin{vmatrix} m_{12} m_{13} \\ m_{22} m_{23} \end{vmatrix} - m_{41}\begin{vmatrix} m_{12} m_{14} \\ m_{22} m_{24} \end{vmatrix} - m_{55}\begin{vmatrix} m_{11} m_{14} \\ m_{41} m_{44} \end{vmatrix} - (m_{44} + m_{66})\begin{vmatrix} m_{11} m_{13} \\ m_{31} m_{33} \end{vmatrix} - (m_{11} + m_{22} + m_{44} + m_{66})\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix} - (m_{11} + m_{22} + m_{33} + m_{55})\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix} - (m_{33} + m_{44} + m_{55} + m_{66})\begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix}$$

$$B_4 = m_{12}\left\{m_{26} m_{61}(m_{33} + m_{55}) + m_{25} m_{51}(m_{44} + m_{66}) + m_{31}\begin{vmatrix} m_{23} m_{25} \\ m_{53} m_{55} \end{vmatrix} + m_{41}\begin{vmatrix} m_{24} m_{26} \\ m_{64} m_{66} \end{vmatrix} - m_{51}\begin{vmatrix} m_{23} m_{25} \\ m_{33} m_{66} \end{vmatrix} - m_{61}\begin{vmatrix} m_{24} m_{26} \\ m_{44} m_{46} \end{vmatrix}\right\} + m_{31}(m_{44} + m_{66})\begin{vmatrix} m_{12} m_{13} \\ m_{22} m_{23} \end{vmatrix} + m_{41}(m_{33} + m_{35})\begin{vmatrix} m_{12} m_{14} \\ m_{22} m_{24} \end{vmatrix} - m_{13}(m_{22} + m_{44} + m_{66})\begin{vmatrix} m_{31} m_{35} \\ m_{51} m_{55} \end{vmatrix} - m_{14}(m_{22} + m_{33} + m_{55})\begin{vmatrix} m_{41} m_{46} \\ m_{61} m_{66} \end{vmatrix} + \left\{(m_{33} + m_{55})(m_{44} + m_{66}) + \begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix} + \begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}\right\}\begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix} + \left\{m_{11} m_{66} + m_{22}(m_{44} + m_{66}) + \begin{vmatrix} m_{11} m_{14} \\ m_{41} m_{44} \end{vmatrix} + \begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}\right\}\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix} + \left\{m_{11} m_{55} + m_{22}(m_{33} + m_{55}) + \begin{vmatrix} m_{11} m_{13} \\ m_{31} m_{33} \end{vmatrix}\right\}\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}$$

$$B_5 = (m_{33} + m_{55})\left\{m_{14} m_{22}\begin{vmatrix} m_{41} m_{46} \\ m_{61} m_{66} \end{vmatrix} + m_{61} m_{12}\begin{vmatrix} m_{24} m_{26} \\ m_{44} m_{46} \end{vmatrix} - m_{12} m_{41}\begin{vmatrix} m_{24} m_{26} \\ m_{64} m_{66} \end{vmatrix}\right\} + (m_{44} + m_{66})\left\{m_{12} m_{51}\begin{vmatrix} m_{23} m_{25} \\ m_{33} m_{35} \end{vmatrix} - m_{12} m_{31}\begin{vmatrix} m_{23} m_{25} \\ m_{53} m_{55} \end{vmatrix}\right\} + m_{13}\left\{m_{22}(m_{44} + m_{66}) + \begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}\right\}\begin{vmatrix} m_{31} m_{35} \\ m_{51} m_{55} \end{vmatrix} - \left\{m_{12} m_{25} m_{51} + m_{31}\begin{vmatrix} m_{12} m_{13} \\ m_{22} m_{23} \end{vmatrix}\right\}\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix} - \left\{(m_{44} + m_{66})\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix} + (m_{33} + m_{55})\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix}\right\}\begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix} - \left\{(m_{11} + m_{22})\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix} + m_{12} m_{26} m_{61} + m_{41}\begin{vmatrix} m_{12} m_{14} \\ m_{22} m_{24} \end{vmatrix} - m_{14}\begin{vmatrix} m_{41} m_{46} \\ m_{61} m_{66} \end{vmatrix}\right\}\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix}$$

$$B_6 = \begin{vmatrix} m_{11} m_{12} \\ m_{21} m_{22} \end{vmatrix}\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix}\begin{vmatrix} m_{44} m_{46} \\ m_{64} m_{66} \end{vmatrix} + \begin{vmatrix} m_{12} m_{14} \\ m_{22} m_{24} \end{vmatrix}\begin{vmatrix} m_{33} m_{35} \\ m_{53} m_{55} \end{vmatrix}\begin{vmatrix} m_{41} m_{46} \\ m_{61} m_{66} \end{vmatrix}$$

$$+\begin{vmatrix} m_{31}m_{35} \\ m_{51}m_{55} \end{vmatrix} \begin{vmatrix} m_{12}m_{13} \\ m_{22}m_{23} \end{vmatrix} \begin{vmatrix} m_{44}m_{46} \\ m_{64}m_{66} \end{vmatrix} - m_{12}m_{26} \begin{vmatrix} m_{41}m_{44} \\ m_{61}m_{64} \end{vmatrix} \begin{vmatrix} m_{33}m_{35} \\ m_{53}m_{55} \end{vmatrix} - m_{12}m_{25} \begin{vmatrix} m_{31}m_{33} \\ m_{51}m_{53} \end{vmatrix} \begin{vmatrix} m_{44}m_{46} \\ m_{64}m_{66} \end{vmatrix}.$$

Where, $\begin{vmatrix} m_1 m_2 \\ m_3 m_4 \end{vmatrix} = m_1 m_4 - m_2 m_3$

The steady rate of the system depends upon the value of the eigenvalues or the characteristic roots. According to Routh–Hurwitz Criterion, the dynamical system remains asymptotically stable if all the obtain roots are either negative or imaginary roots containing negative real parts and it happens if and only if it satisfied all the conditions given below:-

i. $B_j > 0,\ j = 1, \cdots, 6 \quad (1.13a)$

ii. $H_2 = B_1 B_2 - B_3 > 0 \quad (1.13b)$

iii. $H_3 = B_1 B_2 B_3 + B_1 B_5 - B_3^2 - B_1^2 B_4 > 0 \quad (1.13c)$

iv. $H_4 = (B_1 B_2 - B_3)(B_3 B_4 - B_2 B_5) + (B_1 B_2 - B_3) B_1 B_6 - (B_1 B_4 - B_5)^2 > 0 \quad (1.13d)$

v. $H_5 = B_3 (B_1 B_2 - B_3)(B_4 B_5 - B_3 B_6) + B_3 B_5 (B_2 B_5 - B_1 B_6) + B_1 B_3 B_6 (B_1 B_4 - 2B_5) - B_1 (B_2 B_5 - B_1 B_6)^2 - B_5 (B_1 B_4 - B_5)^2 > 0 \ (1.13e)$

The Routh-Hurwitz Criterion are satisfied for the jacobian matrix M_3 at equilibrium E_3 therefore the dynamical system is said to have the existence of global stability i.e. the system is asymptotically stable as t→ ∞ or trajectory movement of the system is towards the equilibrium point .

3.3 Global stability conditions and analysis at the equilibrium points

Now, we obtained the stability of the system globally to enhance the future scope for long-term dynamical behavior of the biological system solution, initiates from a distance position from the fixed point (La-Salle and Lefschetz 1961). Lyapunov's direct method sets few conditions under which the dynamical systems have the existence of global stability in the region of attraction. Hence, when the system dislocates from its origin by a larger distance all the trajectory paths of the line moves towards the stability condition and maintain asymptotically global stability. The fundamental conception of Lyapunov's method provides the region of attraction or basin of attractor and establishes the condition under which the dynamical system is supposed to be asymptotically global stable. The system is said to be globally stable if it must be satisfied all the conditions and the entire trajectory move towards the equilibrium point in the region of attraction under the certain conditions as time tends to infinity. Since the dynamical system is satisfying all the conditions, therefore, the system is globally stable.

Theorem 3.1: In addition to the assumptions, let $K\ (T)$ satisfies the inequalities in the attractor or region of attraction Ω:

$$K_m \leq K(T_1, T_2) \leq K_0, 0 \leq -\frac{\partial K(T_1, T_2)}{\partial T_1} \leq \kappa_1, 0 \leq -\frac{\partial K(T_1, T_2)}{\partial T_2} \leq \kappa_2$$

Here, K_m, K_0, κ_1 & κ_2 are positive constants.

If all the inequalities are satisfied by the proposed system eq. (1.2) then it is globally asymptotically stable at the fixed point E_3 under the certain condition's given below:

$$\left[r_1 U_1^* + r_2 U_2^* - (\alpha + b)\left\{1 - \frac{1}{K(T_1^*, T_2^*)}\right\} - \frac{r N_d^*}{K(T_1^*, T_2^*)}\right]^2 < \frac{4}{25} \frac{r}{K(T_1^*, T_2^*)} \left[r_1 U_1^* + r_2 U_2^* + (\alpha + d) + \frac{r N^*}{K(T_1^*, T_2^*)} - \frac{\alpha + b}{K(T_1^*, T_2^*)} \left\{\frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]} + N_d^*\right\}\right] \quad (1.14a)$$

$$\left[(\pi_1 \nu_1 - \gamma_1)\frac{(Q_1 + Q_2)}{\delta_m} + \left\{(\alpha + b)\frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]} - r K_0\right\}\frac{\kappa_1}{K_m^2}\right]^2 < \frac{4}{15} \frac{r}{K(T_1^*, T_2^*)} (\delta_1 + \gamma_1 N^*) \quad (1.14b)$$

$$\left[(\pi_1 \nu_1 - \gamma_1)\frac{(Q_1 + Q_2)}{\delta_m} + \left\{(\alpha + b)\frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]} - r K_0\right\}\frac{\kappa_2}{K_m^2}\right]^2 < \frac{4}{15} \frac{r}{K(T_1^*, T_2^*)} (\delta_2 + \gamma_2 N^*) \quad (1.14c)$$

$$\left[(\pi_1 \nu_1 - \gamma_1)\frac{(Q_1 + Q_2)}{\delta_m} + \left\{(\alpha + b)\frac{K_0[(r_1 + r_2)(Q_1 + Q_2) - (\alpha + b)\delta_m]}{[(\alpha + d)\delta_m]} - r K_0\right\}\frac{\kappa_2}{K_m^2}\right]^2 < \frac{4}{15} \frac{r}{K(T_1^*, T_2^*)} (\delta_2 + \gamma_2 N^*) \quad (1.14c)$$

$$\left[(\gamma_1-\nu_1)\frac{(Q_1+Q_2)}{\delta_m}\right]^2<\frac{4}{15}\frac{r}{K(T_1^*,T_2^*)}(\beta_1+\nu_1N^*)\quad (1.14d)$$

$$\left[(\gamma_2-\nu_2)\frac{(Q_1+Q_2)}{\delta_m}\right]^2<\frac{4}{15}\frac{r}{K(T_1^*,T_2^*)}(\beta_2+\nu_2N^*)\quad (1.14e)$$

$$\left[\left((\alpha+b)\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right\}^2-\frac{rN^*K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right)\frac{\kappa_1}{K_m^2}\right]^2$$
$$<\frac{4}{15}(\delta_1+\gamma_1N^*)\left[r_1U_1^*+r_2U_2^*+(\alpha+d)+\frac{rN^*}{K(T_1^*,T_2^*)}\right.$$
$$\left.-\frac{\alpha+b}{K(T_1^*,T_2^*)}\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}+N_d^*\right\}\right]\ (1.14f)$$

$$\left[\left((\alpha+b)\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right\}^2-\frac{rN^*K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right)\frac{\kappa_2}{K_m^2}\right]^2$$
$$<\frac{4}{15}(\delta_2+\gamma_2N^*)\left[r_1U_1^*+r_2U_2^*+(\alpha+d)+\frac{rN^*}{K(T_1^*,T_2^*)}\right.$$
$$\left.-\frac{\alpha+b}{K(T_1^*,T_2^*)}\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}+N_d^*\right\}\right]\quad (1.14g)$$

$$\left[r_1\left(K_0-\frac{rN^*K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right)\right]^2$$
$$<\frac{4}{15}(\beta_1+\nu_1N^*)\left[r_1U_1^*+r_2U_2^*+(\alpha+d)+\frac{rN^*}{K(T_1^*,T_2^*)}\right.$$
$$\left.-\frac{\alpha+b}{K(T_1^*,T_2^*)}\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}+N_d^*\right\}\right]\ (1.14h)$$

$$\left[r_2\left(K_0-\frac{rN^*K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}\right)\right]^2$$
$$<\frac{4}{15}(\beta_2+\nu_2N^*)\left[r_1U_1^*+r_2U_2^*+(\alpha+d)+\frac{rN^*}{K(T_1^*,T_2^*)}\right.$$
$$\left.-\frac{\alpha+b}{K(T_1^*,T_2^*)}\left\{\frac{K_0[(r_1+r_2)(Q_1+Q_2)-(\alpha+b)\delta_m]}{[(\alpha+d)\delta_m]}+N_d^*\right\}\right]\ (1.14i)$$

$$[(\gamma_1+\pi_1\nu_1)N^*]^2<\frac{4}{9}(\delta_1+\gamma_1N^*)(\beta_1+\nu_1N^*)\quad (1.14j)$$

$$[(\gamma_2+\pi_2\nu_2)N^*]^2<\frac{4}{9}(\delta_2+\gamma_2N^*)(\beta_2+\nu_2N^*)\quad (1.14k)$$

If the above inequalities or Lyapunovs criterion are satisfied then the proposed model eq. (1.2) is asymptotically globally stable and the system would be remaining stable moving the entire trajectory towards the equilibrium points.

Proof of the theorem 3.1: Lets we consider a positive definite function, which is always remains positive in the region of attraction for certain perturbation of the system to provide the qualitative behavior of the system itself.

$$V(N,N_d,T_1,T_2,U_1,U_2)=$$
$$=\left\{N-N^*-N^*log\frac{N}{N^*}\right\}+\frac{1}{2}(N_d-N_d^*)^2+\frac{1}{2}(T_1-T_1^*)^2+\frac{1}{2}(U_1-U_1^*)^2+\ \frac{1}{2}(T_2-T_2^*)^2+\frac{1}{2}(U_2-U_2^*)^2$$

Differentiating $V(N,N_d,T_1,T_2,U_1,U_2)$ with respect to time "t" along the system of the solution, we get

$$\frac{dV}{dt}=\frac{1}{N}(N-N^*)\frac{dN}{dt}+(N_d-N_d^*)\frac{dN_d}{dt}+(T_1-T_1^*)\frac{dT_1}{dt}+(T_2-T_2^*)\frac{dT_2}{dt}+(U_1-U_1^*)\frac{dT_1}{dt}+(U_2-U_2^*)\frac{dT_2}{dt}$$

$$\frac{dV}{dt}=\frac{1}{N}\{rN-(\alpha+b)N_d\}\left\{1-\frac{N}{K(T_1,T_2)}\right\}(N-N^*)$$
$$+\left[(r_1U_1+r_2U_2)(N-N_d)-(\alpha+d)N_d-\{rN-(\alpha+b)N_d\}\frac{N_d}{K(T_1,T_2)}\right](N_d-N_d^*)$$
$$+(Q_1-\delta_1T_1-\gamma_1NT_1+\pi_1\nu_1NU_1)(T_1-T_1^*)$$
$$+(Q_2-\delta_2T_2-\gamma_2NT_2+\pi_1\nu_1NU_1)(T_2-T_2^*)$$
$$+(\gamma_1NT_1-\beta_1U_1-\nu_1NU_1)(U_1-U_1^*)$$
$$+(\gamma_2NT_2-\beta_2U_2-\nu_2NU_2)(U_2-U_2^*)$$

$$\frac{dV}{dt}=-\frac{r}{K(T_1^*,T_2^*)}(N-N^*)^2-\left\{(r_1U_1^*+r_2U_2^*)+\frac{rN^*}{K(T_1^*,T_2^*)}+(\alpha+d)-\frac{(\alpha+b)(N_d+N_d^*)}{K(T_1^*,T_2^*)}\right\}(N_d-N_d^*)^2$$

$$-(\delta_1+\gamma_1 N^*)(T_1-T_1^*)^2-(\delta_2+\gamma_2 N^*)(T_2-T_2^*)^2-(\beta_1+\gamma_1 N^*)(U_1-U_1^*)^2-(\beta_2+\gamma_2 N^*)(U_2-U_2^*)^2+$$
$$\left\{-(\alpha+b)+\frac{(\alpha+b)}{K(T_1^*,T_2^*)}+(r_1U_1^*+r_2U_2^*)-\frac{rN_d^*}{K(T_1^*,T_2^*)}\right\}(N-N^*)(N_d-N_d^*)$$
$$+[\{(\alpha+b)N_d-rN\}\eta_1(T_1,T_2)-\gamma_1T_1+\pi_1v_1U_1](N-N^*)(T_1-T_1^*)$$
$$+[\{(\alpha+b)N_d-rN\}\eta_2(T_1^*,T_2)-\gamma_2T_2+\pi_2v_2U_2](N-N^*)(T_2-T_2^*)$$
$$+(\gamma_1T_1-v_1U_1)(N-N^*)(U_1-U_1^*)+(\gamma_2T_2-v_2U_2)(N-N^*)(U_2-U_2^*)$$
$$+[(\alpha+b)N_d^2-rN^*N_d]\eta_1(T_1,T_2)(N_d-N_d^*)(T_1-T_1^*)$$
$$+[(\alpha+b)N_d^2-rN^*N_d]\eta_2(T_1^*,T_2)(N_d-N_d^*)(T_2-T_2^*)$$
$$+r_1(N-N_d)(N_d-N_d^*)(U_1-U_1^*)+r_2(N-N_d)(N_d-N_d^*)(U_2-U_2^*)$$
$$+(\gamma_1+\pi_1v_1)N^*(T_1-T_1^*)(U_1-U_1^*)$$
$$+(\gamma_2+\pi_2v_2)N^*(T_2-T_2^*)(U_2-U_2^*)$$

$$\eta_1(T_1,T_2)=\begin{cases}\dfrac{\frac{1}{K(T_1,T_2)}-\frac{1}{K(T_1^*,T_2)}}{T_1-T_1^*}, & T_1\neq T_1^*\\[2ex] \dfrac{-1}{K^2(T_1^*,T_2)}\dfrac{\partial k(T_1^*,T_2)}{\partial T_1}, & T_1=T_1^*\end{cases} \qquad \&$$

$$\eta_2(T_1{}^*,T_2)=\begin{cases}\dfrac{\frac{1}{K(T_1^*,T_2)}-\frac{1}{K(T_1^*,T_2^*)}}{T_2-T_2^*}, & T_2\neq T_2^*\\[2ex] \dfrac{-1}{K^2(T_1^*,T_2^*)}\dfrac{\partial k(T_p^*,T_s^*)}{\partial T_2}, & T_2=T_2^*\end{cases}$$

By using mean value theorem, the inequality satisfies the following equations:-

$$[\eta_1(T_1,T_2)]\le\frac{k_1}{K_m^2}\ \&\ [\eta_2(T_1^*,T_2)]\le\frac{k_2}{K_m^2}$$

Thus, $\frac{dV}{dt}$ can be written as the sum of the quadratics,

$$\frac{dV}{dt}=-\frac{1}{2}a_{11}(N-N^*)^2+a_{12}(N-N^*)(N_d-N_d^*)-\frac{1}{2}a_{22}(N_d-N_d^*)^2$$
$$-\frac{1}{2}a_{11}(N-N^*)^2+a_{13}(N-N^*)(T_1-T_1^*)-\frac{1}{2}a_{33}(T_1-T_1^*)^2$$
$$-\frac{1}{2}a_{11}(N-N^*)^2+a_{14}(N-N^*)(T_2-T_2^*)-\frac{1}{2}a_{44}(T_2-T_2^*)^2$$
$$-\frac{1}{2}a_{11}(N-N^*)^2+a_{15}(N-N^*)(U_1-U_1^*)-\frac{1}{2}a_{55}(U_1-U_1^*)^2$$
$$-\frac{1}{2}a_{11}(N-N^*)^2+a_{16}(N-N^*)(U_2-U_2^*)-\frac{1}{2}a_{66}(U_2-U_2^*)^2$$
$$-\frac{1}{2}a_{22}(N_d-N_d^*)^2+a_{23}(N_d-N_d^*)(T_1-T_1^*)-\frac{1}{2}a_{33}(T_1-T_1^*)^2$$
$$-\frac{1}{2}a_{22}(N_d-N_d^*)^2+a_{24}(N_d-N_d^*)(T_2-T_2^*)-\frac{1}{2}a_{44}(T_2-T_2^*)$$
$$-\frac{1}{2}a_{22}(N_d-N_d^*)^2+a_{25}(N_d-N_d^*)(U_1-U_1^*)-\frac{1}{2}a_{55}(U_1-U_1^*)^2$$
$$-\frac{1}{2}a_{22}(N_d-N_d^*)^2+a_{26}(N_d-N_d^*)(U_2-U_2^*)-\frac{1}{2}a_{66}(U_2-U_2^*)^2$$
$$-\frac{1}{2}a_{33}(T_1-T_1^*)^2+a_{35}(T_1-T_1^*)(U_1-U_1^*)-\frac{1}{2}a_{5\,5}(U_1-U_1^*)^2$$
$$-\frac{1}{2}a_{44}(T_2-T_2^*)^2+a_{46}(T_2-T_2^*)(U_2-U_2^*)-\frac{1}{2}a_{66}(U_2-U_2^*)^2$$

Where,

$$a_{11}=\frac{2}{5}\frac{r}{K(T_1^*,T_2^*)},\ a_{22}=\frac{2}{5}\left\{(r_1U_1^*+r_2U_2^*)+\frac{rN^*}{K(T_1^*,T_2^*)}+(\alpha+d)-\frac{(\alpha+b)(N_d+N_d^*)}{K(T_1^*,T_2^*)}\right\}$$
$$a_{33}=\frac{2}{3}(\delta_1+\gamma_1N^*),\ a_{44}=\frac{2}{3}(\delta_2+\gamma_2N^*),\ a_{55}=\frac{2}{3}(\beta_1+v_1N^*),\ a_{66}=\frac{2}{3}(\beta_2+v_2N^*)$$
$$a_{12}=\left\{-(\alpha+b)+\frac{(\alpha+b)}{K(T_1^*,T_2^*)}+(r_1U_1^*+r_2U_2^*)-\frac{rN_d^*}{K(T_1^*,T_2^*)}\right\},$$
$$a_{13}=[\{(\alpha+b)N_d-rN\}\eta_1(T_1,T_2)-\gamma_1T_1+\pi_1v_1U_1],$$
$$a_{14}=[\{(\alpha+b)N_d-rN\}\eta_2(T_1^*,T_2)-\gamma_2T_2+\pi_2v_2U_2],\ a_{15}=(\gamma_1T_1-v_1U_1),\ a_{16}=(\gamma_2T_2-v_2U_2)$$
$$a_{23}=[(\alpha+b)N_d^2-rN^*N_d]\eta_1(T_1,T_2),\ a_{24}=[(\alpha+b)N_d^2-rN^*N_d]\eta_2(T_1^*,T_2)$$
$$a_{25}=r_1(N-N_d),\ a_{26}=r_2(N-N_d),\ a_{35}=(\gamma_1+\pi_1v_1)N^*,\ a_{46}=(\gamma_2+\pi_2v_2)N^*$$

Thus, $\frac{dV}{dt}$ will be negative definite provided

$$a_{12}^2 < a_{11}a_{22}, a_{13}^2 < a_{11}a_{33}, a_{14}^2 < a_{11}a_{44}, a_{15}^2 < a_{11}a_{55}, a_{16}^2 < a_{11}a_{66}, a_{23}^2 < a_{22}a_{33}, a_{24}^2 < a_{22}a_{44}, a_{25}^2 < a_{22}a_{55}, a_{26}^2 < a_{22}a_{66}, a_{35}^2 < a_{33}a_{55}, a_{46}^2 < a_{44}a_{66} \quad (1.15)$$

Here, eq. (1.15) ⇒ (1.14a – 1.14k). Hence V is a Lyapunov function, whose domain is Ω under the conditions equations. (1.14a – 1.14k). So, E_3 is globally asymptotically stable in Ω under the conditions equations. (1.14a – 1.14k) hold.cc

4. NUMERICAL SIMULATION

A mathematical model is difficult to understand and predict the behavioral change of complex systems. These models help in the critical task such as forecasting before the event actually take place, maximizing the optimal solution for the behavioral change of the various parameter responsible for controlling the various complexities arising to maintain the system and to characterize system response. Now to optimize the stability in the behavior of the system a proprietary programming language MATLAB is used to produce various kinds of output. Here various simulations have been performing for analyzing the result with the help of a computer by using MATCONT and MATLAB packages. Now we assume, the carrying capability function as

$$K(T_1, T_2) = K_0 - \frac{b_{11}T_1}{1 + b_{12}T_1} - \frac{b_{21}T_2}{1 + b_{22}T_2} \quad (1.16)$$

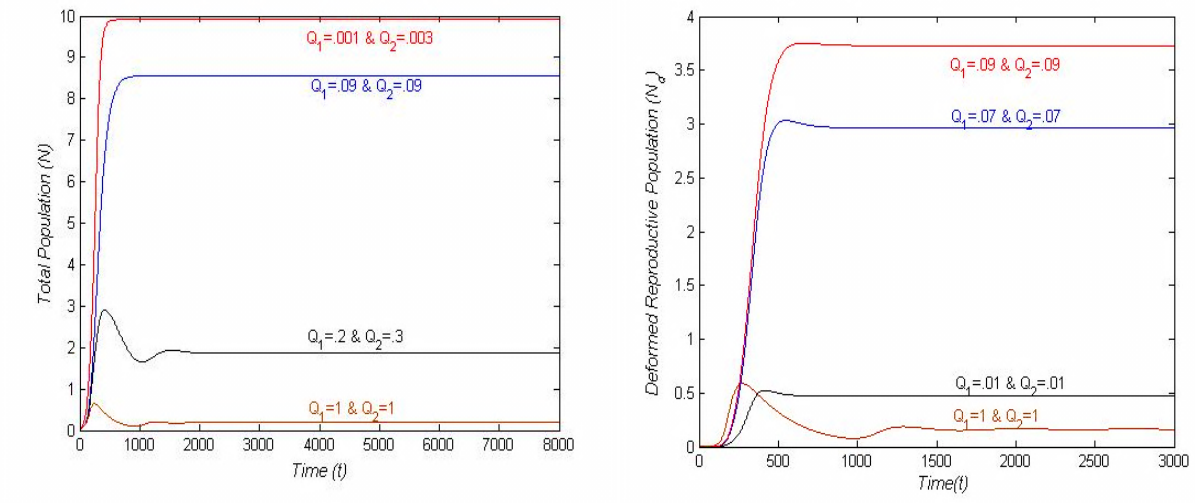
The value of K_m is chosen in such a way that the value lies between $K(T)$ and K_0 i.e. $K(T) \leq K_m \leq K_0$ and a set of the parameters are chosen so that all are constant.

$$K_0 = 10.0, b_{11} = .51, b_{12} = 1.2, b_{21} = 1.3, b_{22} = .5, b = .045, d = .005, r_1 = .05, r_2 = 0.032,$$
$$Q_1 = 0.005, \alpha = .0008, Q_2 = .1, \delta_1 = .005, \delta_2 = 0.01, \pi_1 = .25. \pi_2 = .05, \beta_1 = 1.3, \beta_2 = .1, \gamma_1 = .04,$$
$$\gamma_2 = 0.003, v_1 = .004, v_2 = .003. \quad (1.17)$$

The plot is drawn by modeling the dynamical system for constant values of variables and parameters by keeping one of them varying and then discussed to understand the utility of the proposed dynamical system. This would help to prevent the complete exhaust of the system. Fig.1 (a) the plot shows the simultaneous effect of two reprotoxins on the total population for chosen values of Q_1 & Q_2 and keeping all other parameters constant. The trajectory changes its path continuously at the equilibrium points for the different emission rates of the reprotoxin and becomes more unstable as the rate increases into the environment. This is due to the high impact rate of two increasing reprotoxin at the targeted part resulting in high deformities. Initially, the total population is very high but decreases with the high emission of reprotoxin as a simultaneous effect produces its high effect inside the reproductive parts in a subclass of species Similarly, Fig. 1(b) shows the simultaneous effect of two reprotoxins on the deformed population for chosen values of Q_1 & Q_2 and keeping all other parameters constant. The trajectory changes its path continuously at the equilibrium points for the different emission rates of the reprotoxin. The high emission rate of two increasing reprotoxin at the targeted part resulted in high deformity. The deformed population is very less but increases with high emission of reprotoxin as simultaneous effect produces harmful effect inside the different reproductive parts in a subclass of species. Thus, the simultaneous effect of the reprotoxin increases the infertility rate in a subclass of the species. Fig.2 shows the four case which can occur in the variation of total population for the emission of various values of two reprotoxin. In the first case if $Q_1 = .05$ & $Q_2 = .05$ then the total population is very close to the carrying capacity that is there is less decrease in the total population. In the second case Fig. 2(b) if $Q_1 = .05$ & $Q_2 = .5$, the total population increases to its threshold point and then decreases to its minimum value to attained its steady state. There is less difference in the total population when compared to the first case. In Fig.2(c) if $Q_1 = .5$ & $Q_2 = .05$, the total population decreases to attain a steady state and is much less as compared to the other two cases. This shows that when the value of Q_1 increases the total population increases rapidially and the mortality rate increases. In Fig. 2(d) $Q_1 = .5$ & $Q_2 = .5$, total population firstly rises and then decrease to attain the equilibrium state which is less as compare to the other cases. In the last case if the two reprotoxin increases continuously and emission becomes uncontrolled then the total population decreases have more chances to get extinct from the environment. Fig.3 shows four cases of deformed reproduction population in the biological species with increasing rate of reprotoxin. In the first case, Fig 3(a) if $Q_1 = .05$ & $Q_2 = .05$, the deformed population increases and almost attained the steady state at same level. In second case, Fig.3.(b) the deformed population attains a threshold lower than the first case and attains a steady state, less as compared to the first case. In third case Fig.3 (c), the deformed reproductive population increases to higher compared to the other two cases and attains and become stable at the same point, higher as compared to the other cases. In Fig. 3(d), the deformed reproduction population is very less as compared to the other cases and is less stable. Fig.4 shows that whatever be the mortality rate there is a stable point but this stability decreases with increasing rate of reprotoxin rate. From the above it can be observed as the fixed point decreases the density of the biological species decreases below the carrying capacity with the increasing rate of reprotoxin rate in the environment. Table.1 shows the variation of total population and deformed reproductive population with the increasing rate of two reprotoxins continuously. The various plots on MATLAB suggest that as the emission rate of reproductive toxicant rises and their simultaneous reactions of two reproductive toxicants increases at the site of reproductive part, the loss of reproductive capability increases due to its uptake and this causes the death of individual and the newborn species, due to this the total population decreases below the carrying capacity. Now we plot some 3D-mathematical structure plots based on a synthesized effect of three variables crucial for studying the movement of trajectory path in the region of attraction and helpful in examining the local and global stability after perturbation of the proposed system. These mathematical structures easily visualize the disturbance of the system for the increasing rate of reprotoxin. Out of these three variables, two are independent variables and one is a dependent variable which is representing in matrix form or vector form and the new surface color function is denoted by a new matrix form. The color data is shown by Z-axis also known as a surface height; therefore the color and surface height are both proportional to each other. In this plot, we use a surface with a contour to view mathematical structure over a rectangular grid of regions. Here, the surface with contour creates colored parametric surfaces specified by total population, deformed reproductive population, and two reproductive toxicants, with the color specified by the height of the rectangular grid. Here, three cases have been discussed and analyzed (i) both the reproductive toxicants Q_1 & Q_2 increases together parallel discharged by the external sources continuously, (ii) the reproductive toxicant constant $Q_2 = .05$ and the reproductive toxicant Q_1increases continuously in the environment and (iii) the reproductive toxicant is constant Q_1=.05 and the value of another toxicant Q_2 increases continuously in the environment. Here, all three cases have been plotted using MATLAB. Fig.5, Fig.8 and Fig.11 are the three plots of the total population, deformed reproductive population and two reprotoxins for the three cases in which the two reprotoxins are varying at the different rate from the various external sources.

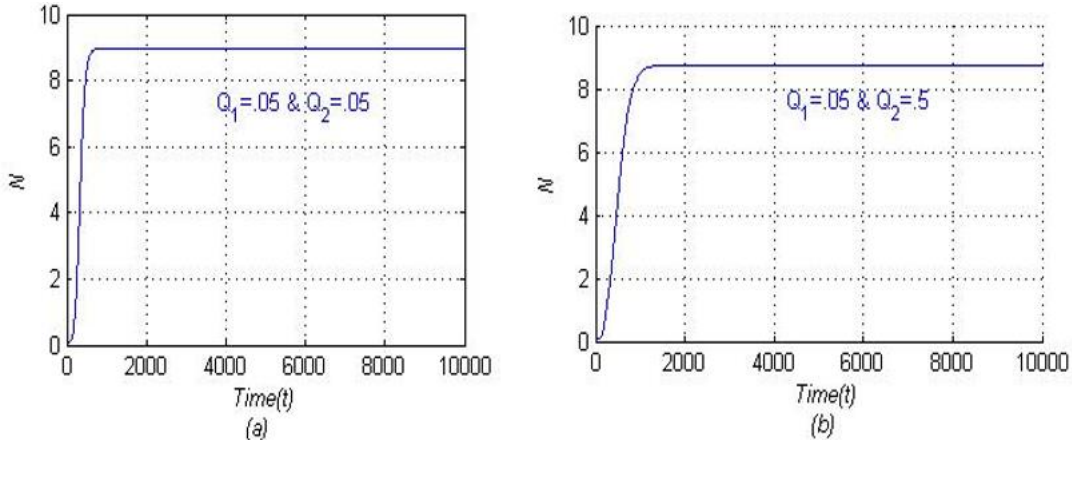
Table.1 N_1, N_2, N_d , T_1, T_2, U_1& U_2 for different value of Q_1 & Q_2.

N	N_d	T_1	T_2	U_1	U_2	Q_1	Q_2
9.9688	0.0486	0.0025	0.0253	0.0007	0.0058	9.9688	0.0486
9.8787	0.1926	0.0101	0.1018	0.0030	0.0233	9.8787	0.1926
9.7937	0.3342	0.0178	0.1793	0.0052	0.0407	9.7937	0.3342
9.7664	0.3808	0.0204	0.2053	0.0059	0.0465	9.7664	0.3808
9.1092	1.7710	0.1090	1.0801	0.0297	0.2318	9.1092	1.7710
8.6392	3.3488	0.2297	2.2440	0.0595	0.4619	8.6392	3.3488
0.9007	0.7866	9.7557	31.4999	0.2696	0.8287	0.9007	0.7866
0.2220	0.1939	64.8447	84.3815	0.4431	0.5583	0.2220	0.1939
0.1874	0.1637	80.0297	94.6784	0.4614	0.5293	0.1874	0.1637



(a) *(b)*

Fig. 1 Variation of N & N_d vs. time (t) for different cases with an increasing rate of reprotoxin.



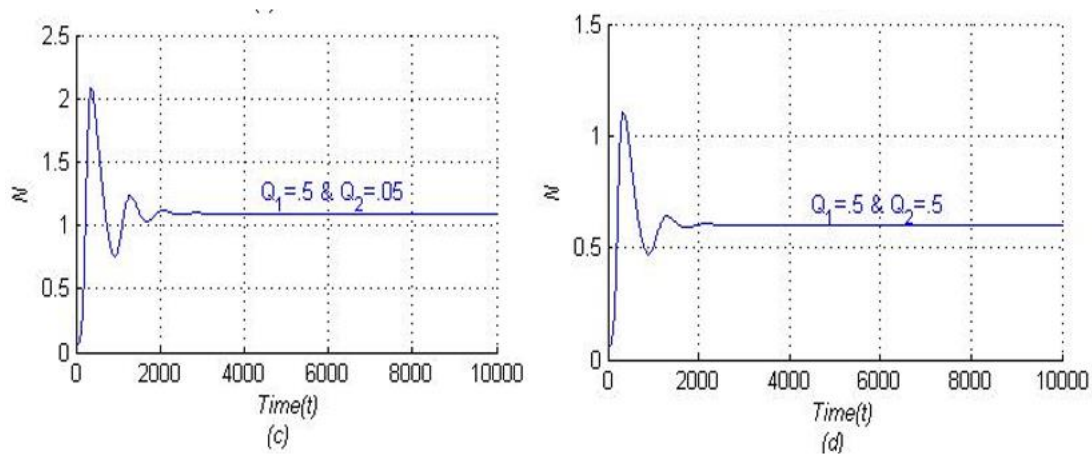


Fig. 2 Total population variation vs. time (t) for different cases with an increasing rate of reprotoxin.

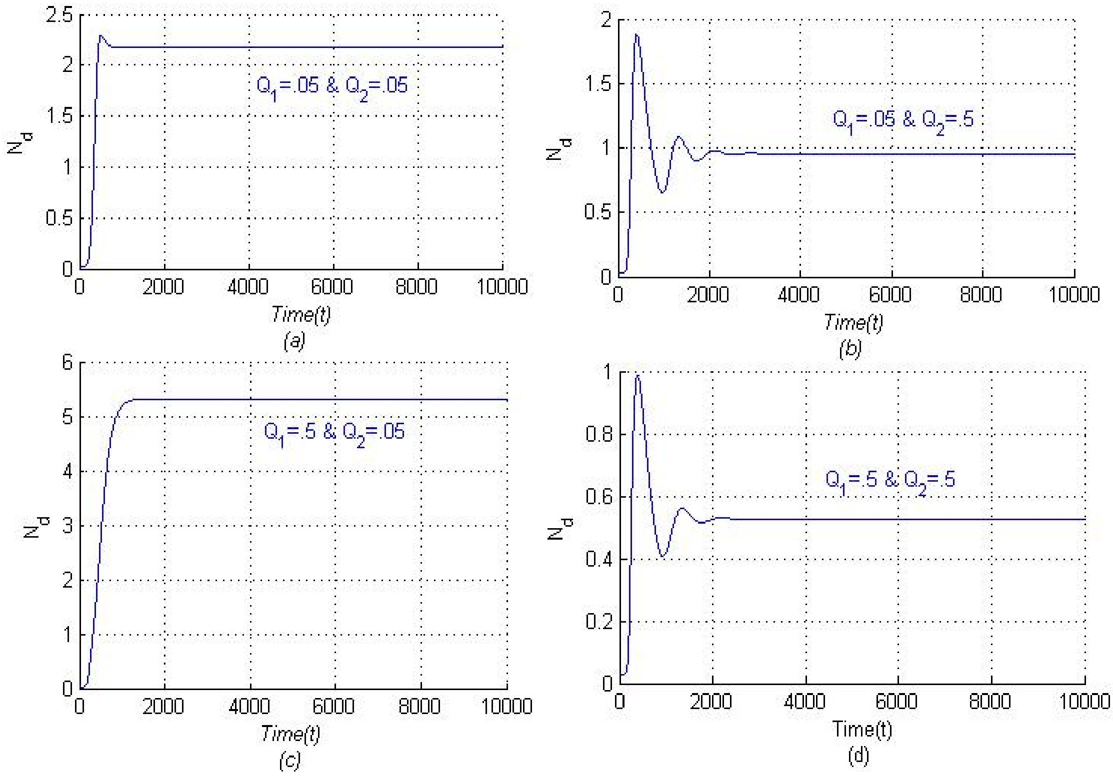


Fig. 3 Variation of deformed reproductive population vs. time (t) for different cases with an increasing rate of reprotoxin.

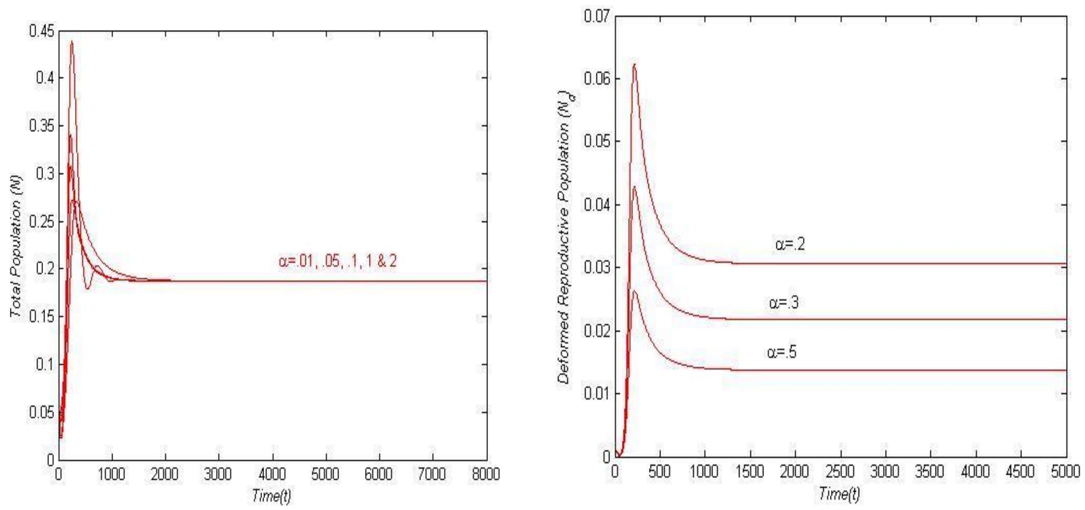


Fig.4 Variation of N & N_d vs. time (t) with increasing mortality rate.

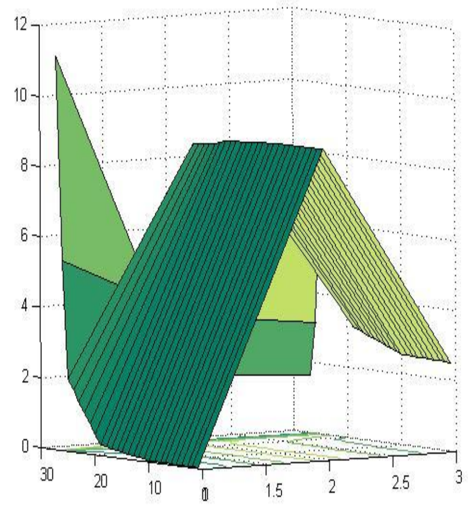


Fig.5 3D surface plot graph with contour of N, N_d & with increasing rate of Q_1 at constant value of Q_2.

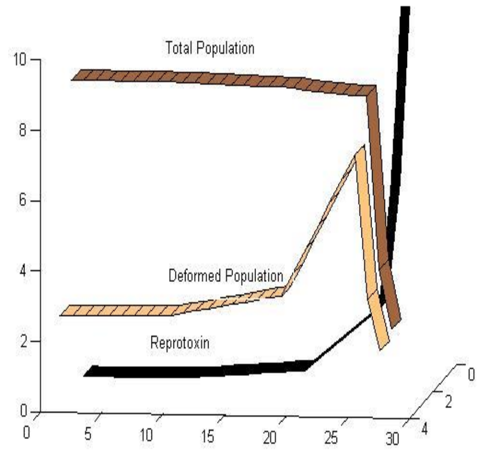


Fig.6 3D Ribbon graph of N, N_d & with increasing rate of Q_1 at $Q_2 = .7$.

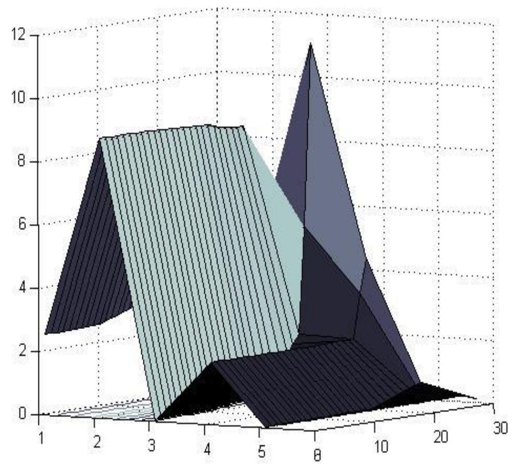


Fig.7 3D Surface plot showing the trajectory movement for the proposed dynamical system with increasing Rate of reprotoxin Q_1.

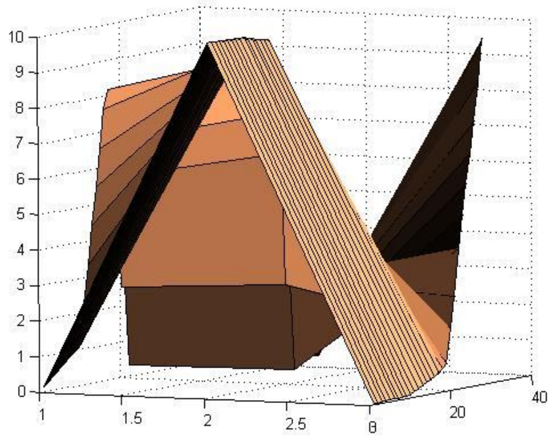


Fig.8 3D surface plot graph with contour of N, N_d & $Q_1 = Q_2$ i.e both are increasing with the same rate.

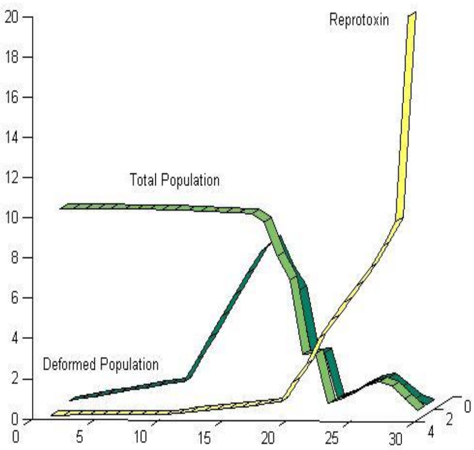


Fig.9 3D Ribbon graph of N, N_d & $Q_1 = Q_2$ movement i.e both are increasing with the same rate.

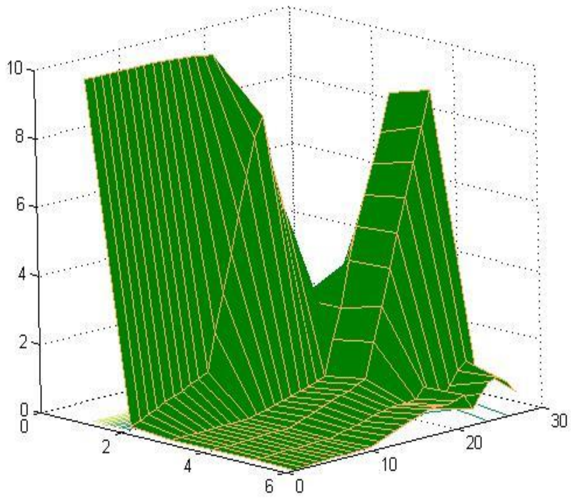
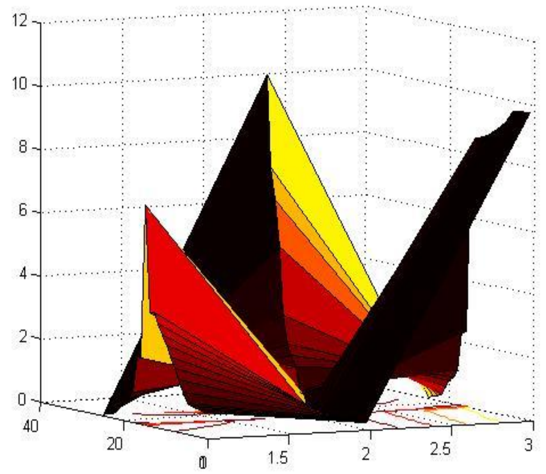


Fig.10 3D Surface plot showing the trajectory for the proposed dynamical system where the both reprotoxin are increasing at the same rate i.e $Q_1 = Q_2$.



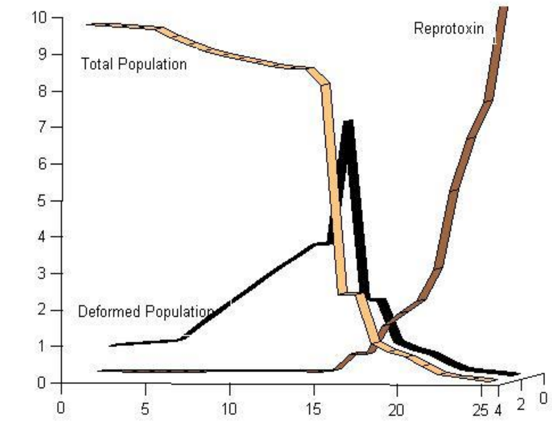
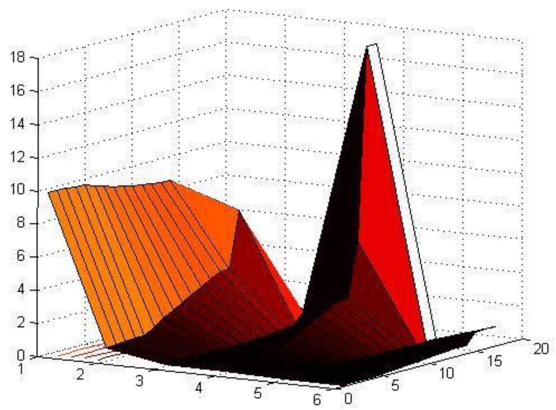


Fig.11 3D Surface plot graph with contour of N, N_d & with increasing rate of Q_2 at constant value of Q_1.

Fig.12 3D Ribbon graph of N, N_d & with increasing rate of Q_1 at $Q_2 = .7$.



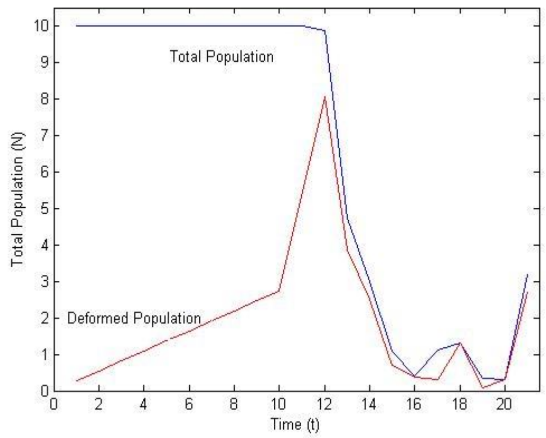


Fig.13 3D Surface plot showing the trajectory movement for the proposed dynamical system with increasing rate of reprotoxin Q_2.

Fig.14 Variation of total population and deformed reproductive population vs. time (t).

Fig.5 is the three-dimensional variable plot or 3D surfaces with the contour of dynamical model eq. (1.2) and the total population, deformed reproductive population and Q_1 is increasing continuously for constant value of Q_2, all are three different variables. This mathematical structure of the dynamical system when Q_1 is increasing continuously for constant value of Q_2. This plot shows that if emission is less i.e. if $Q_2 = 0.7$ & Q_1 =.001 then $N = 8.8184$ & $N_d = 2.5012$ or there is a small decrease in the total population and small increases in the deformed reproductive population. As the reprotoxin increases to $Q_2 = .7$ & $Q_1 = 0.5$ the total population rapidially decreases and the deformed reproductive population increases i.e. $N = 8.5359$ & $N_d = 5.8717$. Similarly when the reprotoxicity level increases to very high i.e. $Q_2 = 0.7$ & Q_1 =.8, both the total population and population of deformities decreases in reproduction and are $N = 3.5905$ & $N_d = 3.1358$ respectively. Fig.8 is the 3D surfaces plot with the contour of dynamical model (1.2) and the total population, deformed reproductive population and $Q_1 = Q_2$, is three variables. This mathematical structure is showing the 3D surface plot of the dynamical system when both the discharging reprotoxin increases into the environment with the same constant rate. This plot shows that if the emission is less i.e. if $Q_1 = Q_2$ = .001 then $N = 9.9688$ & $N_d = 0.0486$ or there is small decrease in the total population and small increases in the deformed reproductive population. As the reprotoxin increases to $Q_1 = Q_2$ = 0.2, the total population rapidially decreases and the deformed reproductive population increases i.e. $N = 4.5395$ & $N_d = 3.9647$. Similarly, when the reprotoxicity level increases to very high i.e. $Q_1 = Q_2$ = .3, the total population decreases and deformed population are $N = 1.5680$ & $N_d = 1.3694$ which is very near to $Q_1 = Q_2$ = 5.2 i.e. $N = 0$ & $N_d = 0$. The maximum malformation in reproductive process in biological species is $N_d = 4.1041$ at $Q_1 = Q_2$ = 0.1. Therefore in the first case it is indicated that if both the reprotoxin increases together the total population doesn't exist for the large emissions of reprotoxin. Fig.11 is the 3D surfaces plot with the contour, three variables i.e. total population, deformed reproductive population and Q_2 is increasing continuously for constant value of Q_1. This mathematical structure is showing the 3D of the dynamical system if Q_2 is increasing continuously for constant value of Q_1 with respect to time (t). This plot shows that if the emission rate is less i.e. if $Q_1 = 0.7$ & Q_2 =.006 then $N = 9.7533$ & $N_d = 0.7913$ or there is small decrease in the total population and small increases in the deformed reproductive population. As the reprotoxin increases to

$Q_1 = 0.7\&$ Q_2 = 0.3 the total population rapidially decreases and the deformed reproductive population increases i.e. $N = 2.3896$ & $N_d = 2.0870$. Similarly when reprotoxicity level increases to very high i.e. $Q_1 = 0.7$ & Q_2 = 4, the total population decreases and equal to $N = .1045$ & $N_d = 0.0912$ which is very near to $Q_1 = 0.7$ & $Q_2 =$ 10 i.e. $N = .0415$ & $N_d = 0.0363$. The maximum malformation in reproductive process in biological species is $N_d = 6.977$ at $Q_1 = 0.7\&$ Q_2 = 0.2. Therefore in the second case it is indicated that if Q_2 is increasing continuously for constant value of Q_1 the total population exist even for the large emissions of reprotoxin. In the third case, the total population density and deformed reproductive population density becomes zero if Q_1 is lying between 5.6 & 6 for constant value of Q_2 i.e. the system is unstable for the large emissions of reprotoxin. In this case the deformities in reproductive process in biological species is maximum at $N_d = 7.2279$ if the reprotoxin increases $Q_1 = .7$ by keeping the other as constant i.e. Q_2 is constant. Therefore in the second case it is indicated that if Q_2 is increasing continuously for constant value of Q_1 the system becomes unstable for the large emissions of reprotoxin i.e. at Q_1 is lying between 5 & 6. From the above it is concluded that, in first case for the emissions of two reprotoxin together, the system remains asymptotically stable for the small emission rate and collapsed at the increasing rate of reprotoxin with time lag. While in second case the system continues to remain stable for long time for increasing rate of reprotoxin and instability increases for very high emission. However, in the third case, the system collapsed for high emissions but was less sensitive compared to the first. It is also observed that if two reprotoxin emitted continuously into the system collapsed sooner as compared to the other two. The failure in the reproductive rate in species gets maximize if the emissions of the two reprotoxin are very high. Fig.6, Fig.9 and Fig.12 are the 3D Ribbon graph structure including three variables, total population, deformed reproductive population, and two reprotoxin increasing continuously for the different cases. Similarly, the proposed dynamical system for the three different cases are represented in Fig.7, Fig.10 and Fig.13, 3D surfaces plot with the contour where the total population, deformed reproductive population, and Q_2 are the three variables. Fig.14 is a plot showing the variation between N and N_d, so that there is a relative change in the total population density and the deformed reproductive population with respect to time. In the diagram, both the total population density and deformed reproductive population increase to their threshold point. As the simultaneous effect of two reproductive toxicants increases at the targeted reproductive organ, both chemical reactivity and structural changes increases. This increases the deformity rate in the subclass as it is generally in the form of a combined, antagonistic, and synergistic state. The infertility rate increases, increasing the mortality rate in biological species and the density of the total population decreases below the carrying capacity. As both the population density becomes negligible in the environment then reprotoxin from the external sources decreases making the system more favorable for the species and raising again the biological species. It concludes that if both reproductive toxicants increases together at the same rate parallel to one another into the environment beyond the admissible level, the total population and the deformed reproductive population both decreases rapidially and hence it increases the probability of extinction of species from the ecosystem imbalancing the nature as all species are depending upon one another to meet their demand. The effects of reprotoxin on the reproduction process or other parts of the body are mediated at many different physiological levels.

RESULT AND DISCUSSION

Population growth, rapid industrial and technological development, urbanization, and injudicious planning without due regard to sustainable development, there have been induced a variety of changes in the environment. A variety of changes are introduce in the environment due to the population growth, technological development, injudicious planning without due regard to sustainable development and rapid industrialization. Abdominal edema or reproduction deformities, liver necrosis, liver atrophy, and abnormal blood smears were also seen in the aquatic habitat. In research, it's also seen that the mortality rate increases in birds due to reproductive defects; the reproduction system is impaired due to decline figures of breeding birds and reducing healthiness of remaining grown-ups. These reproductive toxicants produce an adverse effect on the reproductive process, initially sub-lethal and after getting lethal in biological species as well as yield progressing reprotoxicity in the generation. With discharges of numerous contaminants, reproduction retrieved in some species. Developmental deformities such as embryo mortality and edema have been associated with dioxin in several avian species. Reproductive and physiological differences are companied with population effects in bald eagles and Caspian terns that feed on contaminated fish. The reprotoxin produced by the species is producing additional fecundity in the variety as compared to the external sources. Correspondingly, if we considered the relationship between the population of India and the growing pollution either it would notice that with the growing pollution the fertility rate of India is depleting. Therefore, it's concluded that the infertility rate and the pollution of the climate are directly commensurable to each other. Common problems of infertility in birth species are blocked fallopian tubes due to growing deformities inside them, endometriosis, (PCOS) a hormone imbalance problem, polycystic ovarian syndrome, and primary ovarian insufficiency (POI) ovulation causing reproductive failure, uterine fibroids, which are non-cancerous clumps of tissue and muscle on the walls of the uterus. The majority of the examination has been done on the terrestrial ecosystem with the help of a precise model to show the effect of one or two fresh personal effects of contagious discharging into the environment either from external sources or by the species itself along with their simultaneous effect. Here, three cases have been discussed above and plotted, all have been found stable in the space moving the entire trajectory towards the equilibrium point. But when both increases together and continuously form, their harmful effect on the reproductive process of the natural species yields more harmful effect due to which their fertility decreases and total population decline below the carrying capacity. So, the first situation is more lethal for the biological species compared to the others and is likely to extinct the species from the ecosystem. Hence, if the reprotoxin are emitted with constant and continuous rates, the species in the habitat increases in its infertility rate. So to protect the various endangered species effective mechanisms require to be enforced to regulate and manage the environment for the increasing emission of reprotoxin. It's also observed that in the case of uncontrolled discharges of the two reprotoxins, the species may be doomed to extinction earlier than the case of a single reprotoxin.

CONCLUSION

Reproductive toxicity is a worldwide problem critically affecting the fertility quality in a subclass of the species. This paper is discussed and analyzed for a nonlinear differential model in which the subclass of species gets severely affected by two reproductive toxicants. As infertility is an emergent issue, so the paper is aimed to examine infertility in biological species including human beings. The magnitude of the effect of reprotoxin depends on the reprotoxicity rate and washout quality of each of the reprotoxins. This continuous discharge of reproductive toxicants decreases the reproductive capabilities of various species below the carrying capacity. It observed that the effect of a reprotoxin increased with the growing demand of the biological species decreases the growth rate of the biological species and the other production process. The fertility rate in a subclass of a particular species decrease and is accompanied by the rising reprotoxin discharged into the environment by various external sources. Various plot studies on MATLAB a computer software including the mathematical structure stability suggests that if the discharge of the reproductive toxicants continuously happens for a long time it produces abnormalities in the reproductive organ due to which the reproduction process gets affected at one or further steps leading to declining in the total population which goes beneath the carrying capacity. It has been suggested that there's no safe level of atmosphere toxicants especially when the reproduction process is involved and many reviews have emphasized the adverse effects of environmental toxicants. There's no question that various MATLAB works dynamically suggest that environmental fertilizers do have adverse effects on developmental and reproductive processes. As the reproductive fertilizer rate raises sometime its uptake rate in species rises because of which the reproductive system of species gets affected, affecting the reproduction process in species. This hence decreases the total population density below the carrying capacity. The adverse effects of toxicants on the species have been reviewed the laws and new regulations to stop the adverse effects on biological species and unforeseen side harmful effects of toxicant within the environment be observed and modified. It's establish further through the model study that if suitable efforts are made to reduce the discharge rate of each of the reprotoxin at the source and its absorption in the environment by some removal mechanism, an applicable grade of class consistency can be conserved.

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