

An Analysis Of A Two Predator One Prey Model With Predator Advantage

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Abstract

In this investigation, a mathematical model of a three-way interaction between two predators and one prey is constructed. When it comes to finding prey, these two types of predators are in direct conflict with one another. In addition to this, we are operating under the assumption that one of the predators has a tactical advantage over the other. Analytical and numerical calculations are performed on each and every equilibrium point that is viable to living organisms. The primary mathematical aspects of the model have been examined to a significant degree. In order to corroborate and illustrate the conclusions drawn from the analysis, numerical simulations are often carried out.

Keywords: Predator-prey, Inter-specific competition, Predator advantage, Functional response, Stability

1 INTRODUCTION

A significant amount of enthusiasm for the population interaction model has been voiced by a number of different researchers. The study of how different species interact with one another is one of the subfields of ecology that has received the most attention. The study of dynamical behaviour and interactions between prey and predator is an essential part of both applied mathematics and theoretical ecology. [Case in point:] [Case in point:] In spite of the significant amount of work that has been put into this field, there is still a great deal of ecological and modelling issues that need to be tackled. Because of this, the mathematical analysis of prey-predator systems requires the combined efforts of ecologists and mathematicians. Numerous ecological models have been developed, and we now have a much better understanding of how prey and predator relationships function as a direct result of these models. Ecological modelling is a tool that can be used to study the dynamic behaviour of interacting organisms in an environment [1]. The study of the dynamics of predator-prey systems can be traced back to A.J. Lotka and V. Volterra's writings [2] [3]. The Lotka-Volterra model is the most fundamental model, and it describes the interaction of two species through the use of a set of non-linear coupled first-order ordinary differential equations. In order to provide an explanation for the myriad of ecological phenomena, numerous extensions have been proposed. A new kind of prey or a new kind of predator has been added as a result of one of the changes that have been made. For instance, some researchers have developed mathematical models that represent the interaction between one prey and two predators. These same researchers have also conducted research on dynamic processes, such as the stability of equilibrium states, in order to determine the conditions necessary for coexistence in ecosystems, the extinction of species, the presence of stable periodic solutions, and other related topics [4–8]. In addition, there are a lot of mathematical models that show how two different prey interact with one single predator. The interaction between two species of predators and two kinds of prey was another topic that was investigated by a number of researchers (9).

The term "competition" refers to the circumstance in which two populations that use the same restricted resource compete with one another for the resource to ensure their continued existence [5]. Competition for available resources is a common phenomenon in both natural and social settings [7]. Competition between members of the same species is referred to as intraspecific rivalry, whereas competition between individuals of different species is referred to as interspecific competition. When interspecific competition is taken into account, the system that consists of two predators and one prey becomes more complex than the Lotka-Volterra model. B. Mukhopadhyaya [4] investigated a model consisting of two different kinds of predators and a single type of prey, where the different predator species competed with one another. B. Dubey and colleagues [10] developed a model of two predators vying for the same prey and performed an analysis on it. He demonstrated that the function of food conversion coefficients of predators in ratio-dependent models is quite important when it comes to figuring out how stable planer equilibria will be. Researchers led by Dian Savitri and her colleagues [6] investigated the dynamics of a modified Leslie–Gower one prey–two predators model with competition between populations of predators. The mathematical characterization of the interaction between predators and prey, also known as a functional response, is an essential component of modelling the relationship between predators and prey

populations. The rate at which a predator consumes its prey, also known as the functional response of the predator, is an important aspect of the dynamic that exists between a prey animal and the predator that hunts it. This rate is determined by the typical amount of prey that one predator is able to bring down in a given amount of time [1]. We can employ a variety of functional responses depending on the situation. In the year 1959, Holling [11] provided an explanation of the aspects of the different types of predations and outlined three distinct types of functional responses, which are as follows:

1. Holling Type-I functional response-

$$f_1(x) = gx$$

is characterized by a linear increase of intake rate with the amount of food available. Linear increase assumes that food processing or food searching time and other limitations are negligible. Animals just eat what they can get.

2. Holling Type-II functional response-

In 1913, Michaelis and Menten introduced a non-linear Holling type II response function to study the enzymatic reactions

$$f_2(x) = \frac{jx}{h+x}$$

f_2 denotes intake rate and x denotes resource density. j , h are positive and stands for maximal growth rate and half saturation constant of the species respectively

It is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food.

Another type of functional response known as sigmoidal functional response-

$$f_3(x) = \frac{ix^2}{h+lx+mx^2}$$

3. Holling Type-III functional response-

$$f_4(x) = \frac{ix^2}{h+mx^2}$$

which is the simplified form of $f_3(x)$.

Even though there are a greater number of distinct types of functional responses that can be found. [12–14,23]. In this model, we are going to investigate the dynamics of the predator-prey system by using the Holling type-I functional response. This response assumes that there is a linear increase in intake rate with food density, either for all food densities or only for food densities up to a maximum, after which the intake rate remains constant. The linear rise makes the assumption that the amount of time needed by the consumer to process a food item is insignificant, or that the act of eating does not interfere with the process of hunting for food. A modified Leslie–Gower one prey–two predators' model with the rivalry between predator populations was researched by Dian Savitri et. al. [5], who used Holling type-I functional response for both of the predators. Sahabuddin Sarwardi [15] conducted research on a three-component model that consisted of one population of prey, two populations of predators, and a Holling type II response function that incorporated a constant proportion of prey refuge. This model was considered with a constant proportion of prey refuge. D. Didiharyono [16] discussed the stability of one prey and two predators with the Holling type III and with harvesting at the second predator population. In their article [17], Zhen Wang and colleagues discussed a delayed generalized fractional-order prey–predator model with the interspecific competition. Kalyan Manna and colleagues [18] explored a three-species prey–predator model. In this model, the predator is deemed to be a generalist in nature because it thrives on two different prey species, and there is the intra-specific rivalry between the two prey species. Nijamuddin Ali and colleagues [19] investigated a ratio-dependent food chain model. In this model, the total population was separated into three classes: prey, predator, and top-predator populations. They have also included intra-specific rivalry among predators in the model that they have developed. Some predator species plan their actions in such a way as to provide them an advantage over other predatory animal species. One thing that all of these animals, including lions, piranhas, killer whales, and ants, have in common is that they have all evolved to have the urge to hunt in groups. Hunting in packs is one of the most fascinating animal behaviors because it encompasses such a wide range of tactics that may be utilized by different species in order to successfully capture their prey [24,25]. As a result of social predation, groups of predators are able to discover, target, and kill prey that is larger or more numerous with a greater degree of effectiveness than a single individual is capable of accomplishing on their own. Individual predators can receive numerous fitness advantages through social foraging, such as increased prey detection, the acquisition of more resources, and social or collective hunting, which is considered to have played a crucial part in the evolution of group life. For instance, larger groups of lions (*Panthera leo*) have a better chance of taking down larger prey than smaller groups. [20]. There are a wide variety of animals, such as Barn Owls, Bats, Leopards, and many others, who are dependent on their acute senses in order to hunt and survive in the dim light of the night sky. These nocturnal predators pursue their prey under the cover of darkness, avoiding the competition of daylight, and employ a combination of acute hearing and night vision to locate their targets. This behavior gave them an advantage over other predators since it allowed them to stay hidden during daytime hours when the other predators were active. Depending on what works best for them, spotted hyenas can be seen moving around throughout the day as well as the night.

Following the consideration of a few presumptions, we will investigate a mathematical model of two predators competing for the same prey, with one predator species employing a more fruitful hunting technique as compared to the other.

Within the second section, we present the model that was mentioned earlier.

In the third section, we talk about the positivity and boundedness of the solution, as well as the local stability analysis.

Analysis of the global stability situation is carried out in section 4.

The results of the numerical simulations used to verify the findings are presented in Section 5, which is followed by a discussion of the findings and a conclusion in Section 6.

2 MODEL FORMULATION

Before formulating the model, we make the following assumptions:

1. We consider a model of Two predator species competing for one prey species where s is the population size of the prey species, p_1 and p_2 are the population sizes of the predator species at any time t .
2. We assume that prey growth is logistic in nature in the absence of the predator, owing to limited resources in nature. Where the carrying capacity of the ecosystem is k .
3. Since there are two predator populations, they will eat prey, which may be termed as the interaction of the prey with two predators and the mathematical term dealing with these interactions is called functional responses. There are various types of functional responses that we can use according to our model. Here we use Holling Type-1 functional response for both the predators. The terms sp_1r_2 and sp_2r_3 denote the first and second predators' response to the prey species, respectively.
4. Competition occurs when access to resources is negatively affected by the presence of other individuals. Predator competition can occur among conspecific species and/or heterospecific species sharing prey resources. Intraspecific competition decreases consumption rates within a single species, while interspecific competition decreases consumption rates of other species sharing the same resources. We assume There is competition [4] among the two predators which may be described mathematically in general as $k_1p_1p_2$ and $k_2p_1p_2$.
5. We consider a natural death among the two predator species.

Under the above assumptions, we have the following model equations:

$$\begin{aligned} \frac{ds}{dt} &= r_1s\left(1 - \frac{s}{k}\right) - d_1s - sp_1r_2 - sp_2r_3 \\ \frac{dp_1}{dt} &= r_4r_2sp_1 - k_1p_1p_2 - d_2p_1 \\ \frac{dp_2}{dt} &= r_5sp_2r_3 - k_2p_1p_2 - d_3p_2 \end{aligned} \quad (1)$$

which is studied by Hsu et.al. [21].The authors studied the relationship between the coefficient of interference and competition outcome.

Here, we use logistic growth for the prey in absence of the predator which is given by the term $r_1s\left(1 - \frac{s}{k}\right)$.Here if $s=k$ then this term becomes zero i.e at the saturation point ,there is no growth. We assume that the prey is very much smaller in size as compared to both the predator species and hence the handling time is negligible .So, Holling Type-I functional response is appropriate here.

The first predator(p_1) consumes the prey(s) with Holling Type-I functional response r_2sp_1 and contributes to its growth with rate $r_4r_2sp_1$.Similarly,The 2nd predator(p_2) consumes the prey(s) with Holling Type-I functional response r_3sp_2 and contributes to its growth with rate $r_5r_3sp_2$.

We assumes that one predator(p_2)has an advantageous hunting strategy (e.g.: night hunting,etc) compared to the other predator (p_1) which increases its chances of successful hunting.

The above model has been updated by incorporating a predator advantage in p_2 .Here m is the predator advantage coefficient which is proportional to prey density ms .

To incorporate the predator advantage effect, we multiply the term $(1+m)$ in the functional response of the second predator i.e $r_3sp_2(1 + m)$

Under these additional effects, the above system 1 reduces to the following modified form:

$$\begin{aligned} \frac{ds}{dt} &= r_1s\left(1 - \frac{s}{k}\right) - sp_1r_2 - sp_2r_3(1 + m) \\ \frac{dp_1}{dt} &= r_4r_2sp_1 - k_1p_1p_2 - d_2p_1 \\ \frac{dp_2}{dt} &= r_5sp_2r_3(1 + m) - k_2p_1p_2 - d_3p_2 \end{aligned} \quad (2)$$

with initial conditions:

$$s(0) = s^0 > 0, p_1(0) = p_1^0 > 0, p_2(0) = p_2^0 > 0$$

In the above model, We consider:

3 ANALYSIS OF THE PROPOSED MODEL

In this section, we study positivity and boundedness,existence of equilibrium points of the proposed system (2) and examine their stability.

System parameter	Significance of the parameter
s, p_1, p_2	Population sizes of the prey, first predator and second predator respectively at any time t ,
r_1 and k	Intrinsic growth rate and environmental carrying capacity for the prey respectively
d_2, d_3	Death rate of the prey, first predator and the second predator respectively
k_1, k_2	The interference rate for the first predator and second predator respectively
r_2, r_3	Predation rate for the first predator and the second predator respectively
r_4, r_5	predator's conversion efficiency of the two predator species
m	predator advantage coefficient

Table 1: Significance of system parameters.

3.1 Positivity and Boundedness of the solution:

It is important to show the positivity and the boundedness of the solution of the system (2) as they represent populations. To show that all the solutions of our system (2) are positive at first let us write our present system (2) as follows:

$$\begin{aligned}\frac{dS}{dt} &= S\phi_1(S, P_1, P_2) \\ \frac{dP_1}{dt} &= P_1\phi_2(S, P_1, P_2) \\ \frac{dP_2}{dt} &= P_2\phi_3(S, P_1, P_2)\end{aligned}\quad (3)$$

We now use the following lemma to show that all the solutions of our system are positive.

Lemma 1: Any solution of the differential equation $\frac{dX}{dt} = X\chi(X, Y)$ is always positive.

Theorem 1: Solutions to system (2) are always positive.

Proof: Since system (2) can be written in the form of system 3 then the proof of the theorem follows from Lemma 1.

Theorem 2: The solution of the system (2) exists in \mathbb{R}^3_+ for $t \geq 0$ and it is bounded.

Proof: System (1) is bounded given by [21]. One can easily find that system (2) is also bounded by the same notion.

3.2 Equilibrium analysis:

In this section, we discuss the equilibrium points of system (2) with their existence conditions. Mathematically, there exist six types of equilibrium points but biologically we have only five and they are as follows:

(a) Trivial equilibrium point, $E_0 \equiv (0, 0, 0)$ and it always exist. It is the scenario where no prey as well as no predator exists

The boundary points:

(b) Predators free equilibrium point, $E_1 \equiv (k, 0, 0)$ and it always exists;

It is the scenario where prey attains carrying capacity and there will be no predators.

(c) Boundary equilibrium point which is free from predator P_2 , $E_2 \equiv (B_1, C_1, 0)$ Where,

$$B_1 = \frac{d_2}{r_2 r_4}, C_1 = \frac{r_1(-d_2 + k r_2 r_4)}{k r_2^2 r_4}$$

and it exists under the condition $k > \frac{d_2}{r_2 r_4}$;

It is the scenario which describes extinction of P_2 .

(d) Boundary equilibrium point which is free from predator P_1 , $E_3 \equiv (B_2, 0, D_1)$ Where,

$$B_2 = \frac{d_3}{(1+m)r_3 r_5}, D_1 = \frac{r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)^2 r_3^2 r_5}$$

and it exists under the condition; $k > \frac{d_3}{r_3 r_5 + m r_3 r_5}$

It is the scenario which describes extinction of P_1 .

(e) Prey free equilibrium point, $E_4 \equiv (0, C_2, D_2)$ Where,

$$C_2 = -\frac{d_3}{k_2}, D_2 = -\frac{d_2}{k_1}$$

and it does not exist;

It is the scenario which describes extinction of prey species. But if prey species goes extinct then the system will collapse i.e it is not biologically feasible.

(f) Interior equilibrium point which represents the coexistence of the species, $E^* = (s^*, p_1^*, p_2^*)$

Where,

$$s^* = -\frac{-kk_1k_2r_1-d_3kk_1r_2-d_2kk_2r_3-d_2kk_2mr_3}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

$$p_1^* = -\frac{d_3k_1r_1+d_3kr_2r_3r_4+d_3kmr_2r_3r_4-kk_1r_1r_3r_5-kk_1mr_1r_3r_5-d_2kr_3^2r_5-2d_2kmr_3^2r_5-d_2km^2r_3^2r_5}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

$$p_2^* = -\frac{d_2k_2r_1-kk_2r_1r_2r_4-d_3kr_2^2r_4+d_2kr_2r_3r_5+d_2kmr_2r_3r_5}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

3.3 Local stability analysis of equilibrium points:

The local behavior of system (2) is investigated by considering its linear approximation around each equilibrium state. The variational matrix of the linearized system around a point (s, p_1, p_2) is given by

$$J(s, p_1, p_2) = \begin{bmatrix} -p_1r_2 - (1+m)p_2r_3 - \frac{r_1s}{k} + r_1(1 - \frac{s}{k}) & -r_2s & -(1+m)r_3s \\ p_1r_2r_4 & -d_2 - k_1p_2 + r_2r_4s & -k_1p_1 \\ (1+m)p_2r_3r_5 & -k_2p_2 & -d_3 - k_2p_1 + (1+m)r_3r_5s \end{bmatrix}$$

Now we have the following:

i) The variational matrix around the trivial equilibrium point E_0 is

$$J(E_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}$$

The eigenvalues of the variational matrix $J(E_0)$ are r_1 , $-d_2$ and $-d_3$. Here, one eigenvalue is positive and two are negative. Hence the predator prey system (2) is always unstable around trivial equilibrium point E_0 .

ii) The variational matrix around the predator free equilibrium point E_1 is

$$J(E_1) = \begin{bmatrix} -r_1 & -kr_2 & -k(1+m)r_3 \\ 0 & -d_2 + kr_2r_4 & 0 \\ 0 & 0 & -d_3 + k(1+m)r_3r_5 \end{bmatrix}$$

The eigenvalues of $J(E_1)$ are $-r_1$, $-d_2 + kr_2r_4$, $-d_3 + k(1+m)r_3r_5$. Now, if the conditions $k < \frac{d_2}{r_2r_4}$ and $k < \frac{d_3}{r_3r_5 + r_3mr_5}$ holds, then all three eigenvalues are purely negative. Hence, equilibrium E_1 is locally asymptotically stable if the aforesaid two conditions hold simultaneously. iii) The variational matrix around the predator free equilibrium point E_2 can be calculated as

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where,

$$a_{11} = r_1(1 - \frac{d_2}{kr_2r_4}) - \frac{d_2r_1}{kr_2r_4} - \frac{r_1(-d_2 + kr_2r_4)}{kr_2r_4}$$

$$a_{12} = -\frac{d_2}{r_4}, a_{13} = -\frac{(d_2(1+m)r_3)}{r_2r_4},$$

$$a_{21} = \frac{r_1(-d_2 + kr_2r_4)}{kr_2}, a_{22} = 0, a_{23} = -\frac{k_1r_1(-d_2 + kr_2r_4)}{kr_2^2r_4},$$

$$a_{31} = 0, a_{32} = 0, a_{33} = -d_3 - \frac{k_2r_1(-d_2 + kr_2r_4)}{kr_2^2r_4} + \frac{d_2(1+m)r_3r_5}{r_2r_4}$$

The characteristic equation of the above variational matrix is

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0,$$

where

$$F_1 = -(a_{11} + a_{33}),$$

$$F_2 = a_{11}a_{33} - a_{12}a_{21}$$

$$F_3 = a_{12}a_{21}a_{33}$$

If we choose

$$k > \frac{d_2}{r_2r_4}$$

and

$$k_2 < -\frac{d_2kr_2r_3r_5}{d_2r_1 - kr_1r_2r_4}$$

and

$$d_3 > \frac{d_2k_2r_1 - kk_2r_1r_2r_4 + d_2kr_2r_3r_5}{kr_2^2r_4}$$

and

$$m < \frac{-d_2 k_2 r_1 + k k_2 r_1 r_2 r_4 + d_3 k r_2^2 r_4 - d_2 k r_2 r_3 r_5}{d_2 k r_2 r_3 r_5}$$

then $F_i > 0$ where $i = 1, 2, 3$ and $F_1 F_2 > F_3$. Hence it follows Routh-Hurtwitz criteria of local asymptotic stability. Thus the system (2) is locally asymptotically stable around positive equilibrium E_2 . iv) The variational matrix around the predator free equilibrium point E_3 can be calculated as

$$J(E_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where,

$$b_{11} = r_1 \left(1 - \frac{d_3}{k(1+m)r_3 r_5}\right) - \frac{(d_3 r_1)}{k(1+m)r_3 r_5} - \frac{r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)r_3 r_5}, b_{12} = -\frac{(d_3 r_2)}{(1+m)r_3 r_5}, b_{13} = -\frac{d_3}{r_5}$$

$$b_{21} = 0, b_{22} = -d_2 + \frac{d_3 r_2 r_4}{(1+m)r_3 r_5} - \frac{k_1 r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)^2 r_3^2 r_5}, b_{23} = 0,$$

$$b_{31} = \frac{r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)r_3}, b_{32} = -\frac{(k_2 r_1(-d_3 + k r_3 r_5 + k m r_3 r_5))}{k(1+m)^2 r_3^2 r_5}, b_{33} = 0$$

The characteristic equation of the above variational matrix is

$$\Lambda^3 + H_1 \Lambda^2 + H_2 \Lambda + H_3 = 0,$$

where

$$H_1 = -(b_{11} + b_{22}),$$

$$H_2 = b_{11} b_{22} - b_{13} b_{31}$$

$$H_3 = b_{13} b_{31} b_{22}$$

$$k > \frac{d_3}{r_3 r_5 + m r_3 r_5}$$

If we choose

and

$$r_2 < \frac{-d_3 r_3 r_5 - d_3 m r_3 r_5 + k r_3^2 r_5^2 + 2k m r_3^2 r_5^2 + k m^2 r_3^2 r_5^2}{d_3 r_4}$$

and $r_1 < k r_2 r_4$

$$k_1 > \frac{-d_3 k r_2 r_3 r_4 - d_3 k m r_2 r_3 r_4}{d_3 r_1 - k r_1 r_3 r_5 - k m r_1 r_3 r_5}$$

and

from the above variational matrix, then $H_i > 0$ where $i = 1, 2, 3$ and $H_1 H_2 > H_3$. Hence it follows Routh-Hurtwitz criteria of local asymptotic stability. Thus the system (2) is locally asymptotically stable around positive equilibrium E_3 .

v) The variational matrix around the coexistence equilibrium point E^* can be calculated as

$$J(E^*) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where,

$$c_{11} = \left(1 - \frac{s^*}{k}\right) r_1 - \frac{(s^* r_1)}{k} - p_1^* r_2 - p_2^* (1+m) r_3, c_{12} = -s^* r_2, c_{13} = -s^* (1+m) r_3$$

$$c_{21} = p_1^* r_2 r_4, c_{22} = -d_2 - p_2^* k_1 + s^* r_2 r_4, c_{23} = -p_1^* k_1,$$

$$c_{31} = p_2^* (1+m) r_3 r_5, c_{32} = -p_2^* k_2, c_{33} = -d_3 - p_1^* k_2 + s^* (1+m)$$

The characteristic equation of the above variational matrix is

$$\lambda^3 + G_1 \lambda^2 + G_2 \lambda + G_3 = 0 \tag{4}$$

where

$$G_1 = -(c_{11} + c_{22} + c_{33}),$$

$$G_2 = c_{11} c_{33} + c_{11} c_{22} + c_{22} c_{33} - c_{12} c_{21} - c_{13} c_{31} - c_{23} c_{32}$$

$$G_3 = c_{11}(c_{22} c_{33} - c_{23} c_{32}) + c_{12}(c_{23} c_{31} - c_{21} c_{33}) + c_{13}(c_{21} c_{32} - c_{22} c_{31})$$

By the Routh-Hurwitz criterion, E^* is locally asymptotically stable if and only if

$$G_1 > 0, G_3 > 0 \text{ and } G_1 G_2 - G_3 > 0$$

Theorem:3 The local stability of the equilibrium points of system ((2)) is given by

(i) E_0 is always unstable.

(ii)

holds. E_1 is locally asymptotically stable if conditions $k < \frac{d_2}{r_2 r_1}$ and $k < \frac{d_3}{r_3 r_5 + r_3 m r_5}$ holds.

(iii) The system (2) around predator free equilibrium point E_2 is locally asymptotically stable if conditions

$$k > \frac{d_2}{r_2 r_4}$$

and

$$k_2 < -\frac{d_2 k r_2 r_3 r_5}{d_2 r_1 - k r_1 r_2 r_4}$$

$$d_3 > \frac{d_2 k_2 r_1 - k k_2 r_1 r_2 r_4 + d_2 k r_2 r_3 r_5}{k r_2^2 r_4}$$

and
and

$$m < \frac{-d_2 k_2 r_1 + k k_2 r_1 r_2 r_4 + d_3 k r_2^2 r_4 - d_2 k r_2 r_3 r_5}{d_2 k r_2 r_3 r_5}$$

holds.

(iv) E_3 is locally asymptotically stable if conditions

$$k > \frac{d_3}{r_3 r_5 + m r_3 r_5}$$

and

$$r_2 < \frac{-d_3 r_3 r_5 - d_3 m r_3 r_5 + k r_3^2 r_5^2 + 2 k m r_3^2 r_5^2 + k m^2 r_3^2 r_5^2}{d_3 r_4}$$

and

$$r_1 < k r_2 r_4$$

and

$$k_1 > \frac{-d_3 k r_2 r_3 r_4 - d_3 k m r_2 r_3 r_4}{d_3 r_1 - k r_1 r_3 r_5 - k m r_1 r_3 r_5}$$

holds.

(v) The coexistence equilibrium state E^* is locally asymptotically stable if and only if G_1, G_3 and $G_1 G_2 - G_3$ are positive.

4 GLOBAL STABILITY

4.1 Global Stability of E_1 :

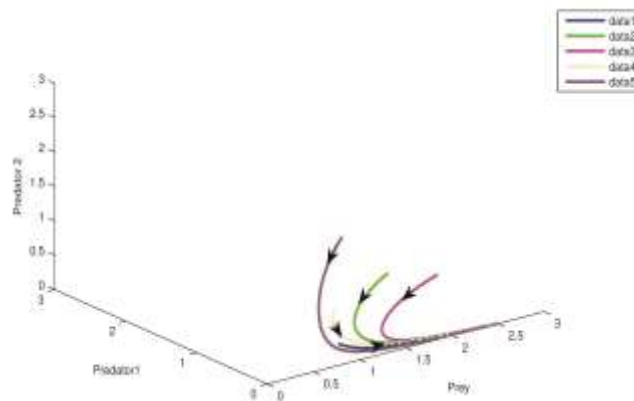


Figure 1: Axial Equilibrium point $E_1=(k,0,0)$ is globally asymptotically stable.

Here, data1=(1,0.3,0.1), data2=(2,0.9,0.5), data3=(2,0.2,0.8), data4=(1.5,1,0.1), data5=(1.5,0.9,1.2)

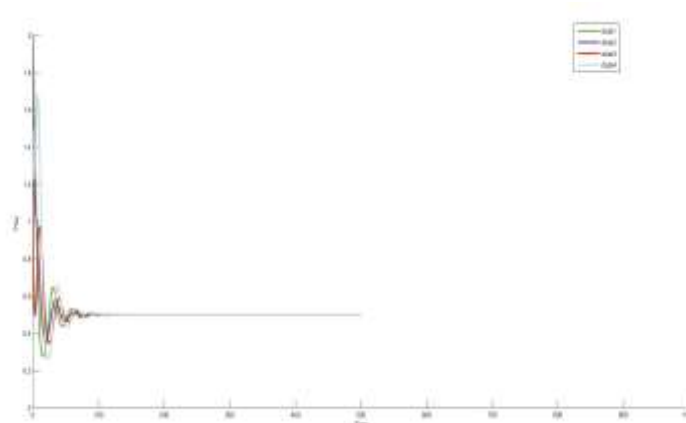


Figure 2: Time series of Prey around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, data1=(1.5,0.9,1.2), data2=(1.5,0.1,0.6), data3=(2,0.3,0.1), data4=(2,0.4,0.5)

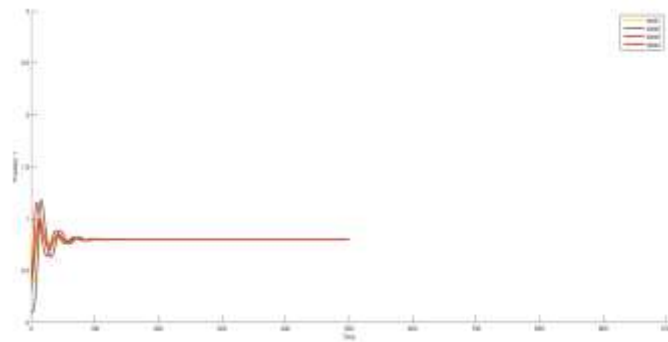


Figure 3: Time series of Predator 1 around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, $data1=(1.5,0.9,1.2), data2=(1.5,0.1,0.6), data3=(2,0.3,0.1), data4=(2,0.4,0.5)$

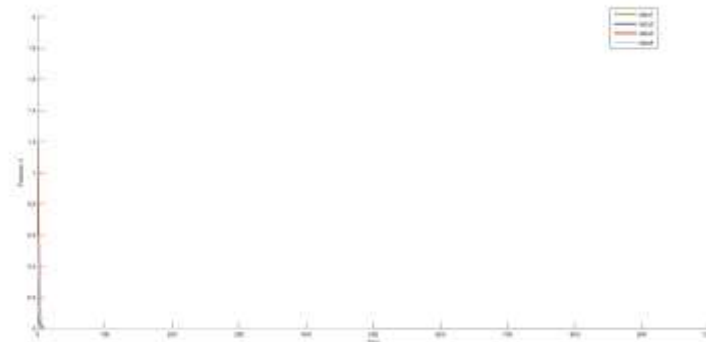


Figure 4: Time series of Predator 2 around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, $data1=(1.5,0.9,1.2), data2=(1.5,0.1,0.6), data3=(2,0.3,0.1), data4=(2,0.4,0.5)$

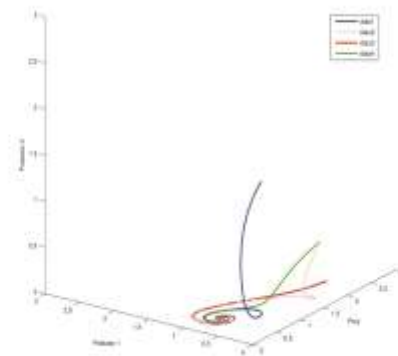


Figure 5: Phase portrait around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, $data1=(1.5,0.9,1.2), data2=(1.5,0.1,0.6), data3=(2,0.3,0.1), data4=(2,0.4,0.5)$

5 NUMERICAL VERIFICATION:

The main aim of this section is to verify the theoretical results developed in the previous sections with the help of MATHEMATICA software as well as MATLAB software. Real world data are not available for this model. Therefore, we use simulated data to illustrate the results obtained from our theoretical analysis.

Some hypothetical biotically feasible values of parameter have been considered as shown in the following table:

Parameter	value
r_1	0.5
r_2	0.5
r_3	0.5
r_4	0.5
r_5	0.25
d_3	0.7343
k_1	0.5
k	2.5

5.1 Equilibrium points:

To verify the global stability of the predators free equilibrium, we take parameter values as follows: $m = 0.5, d_2 = 0.8125, k_2 = 0.5$ and the rest are from the above table. Many initial values of co-extant population have taken as shown in figure which converges to the axial equilibrium point which confirms the global stability and in turns confirms the local stability as shown in figure (1). However as the value of d_2 and k_2 decreases and when $d_2=0.125, k_2=0.002$ with all the other parameter values taking the same as above, E_1 loses its stability and E_2 found to be existant as well as stable by Routh-Hurtwitz criteria as shown in figure (2,3,4,5). On the other hand, keeping the value of d_2 and k_2 fixed as $d_2 = 0.8125, k_2 = 0.5$ and then as the value of m increases $m=3.8744, E_1$ loses its stability and E_3 found to be existant as well as stable by Routh-Hurtwitz criteria as shown in figure (6,7,8,9). Also, using MATHEMATICA software, we have found that for no combination of parameter values, the stability conditions of the co-existing equilibrium point E^* satisfies. Hence, the result that E^* is unstable.

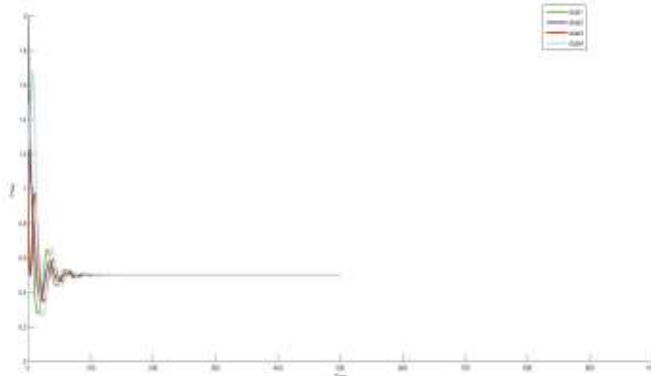


Figure 6: Time series of Prey around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, $data1=(2,0.3,0.5), data2=(2,0.1,0.8), data3=(1.5,0.5,1), data4=(1.5,0.9,0.8)$

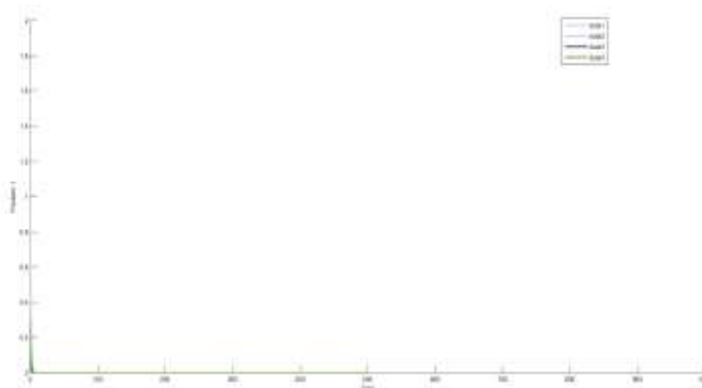


Figure 7: Time series of Predator 1 around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, $data1=(2,0.3,0.5), data2=(2,0.1,0.8), data3=(1.5,0.5,1), data4=(1.5,0.9,0.8)$

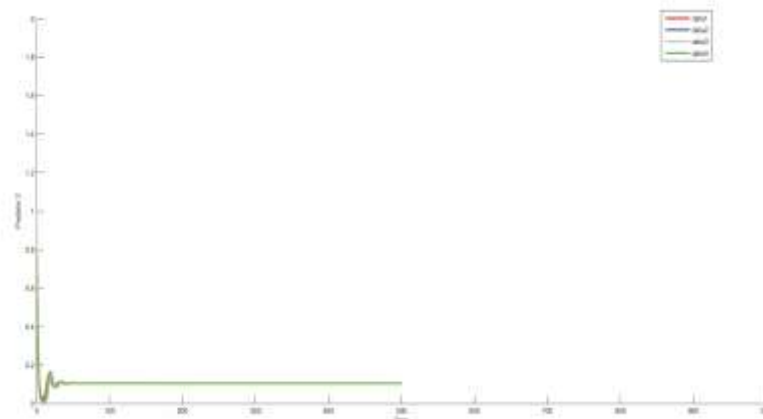


Figure 8: Time series of Predator 2 around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, $data1=(2,0.3,0.5), data2=(2,0.1,0.8), data3=(1.5,0.5,1), data4=(1.5,0.9,0.8)$

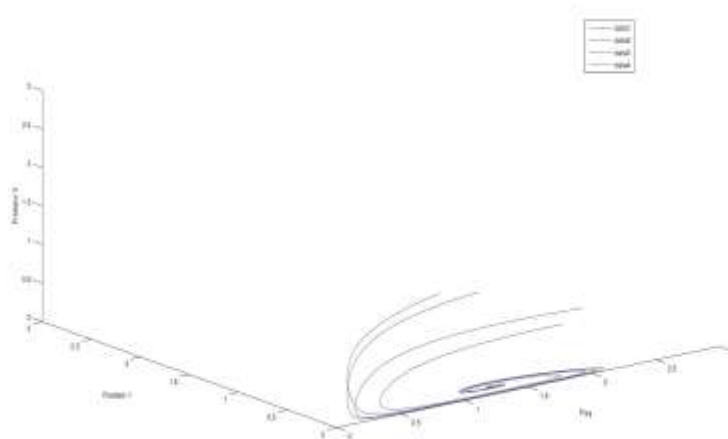


Figure 9: Phase portrait around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, $data1 = (2, 0.3, 0.5), data2 = (2, 0.1, 0.8), data3 = (1.5, 0.5, 1), data4 = (1.5, 0.9, 0.8)$

5.2 Existence of Bistability :

Taking parameter values as $r_1 = 0.125, r_2 = 0.5, r_3 = 0.5, r_4 = 0.03125, r_5 = 0.0625, m = 0.25, d_2 = 0.015625, d_3 = 0.0625, k_1 = 0.5, k_2 = 0.5, k = 2$, it is found that it satisfies the Routh-Hurwitz criteria for E_2 as well as E_3 . Thus, for these parameter values both E_2 and E_3 found to be existent as well as stable. Hence bistability as shown in figure (10).

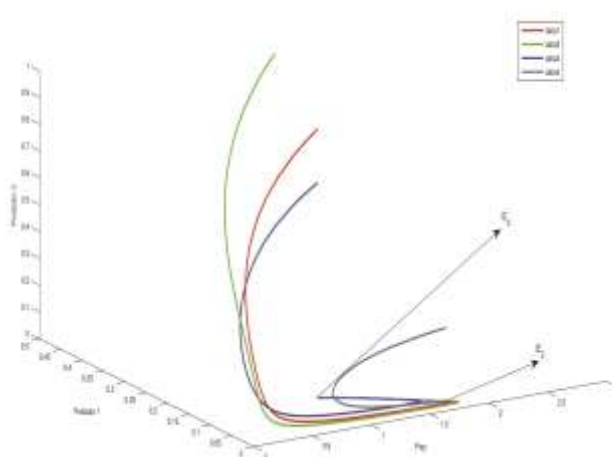


Figure 10: Existence of bistability showing existence of E_2 as well as E_3 simultaneously. Here, $data1 = (2, 0.4, 0.7), data2 = (2, 0.5, 0.9), data3 = (2, 0.4, 0.5), data4 = (2, 0.1, 0.2)$

6 DISCUSSION AND RESULT:

In this paper, we consider a two predator one prey biosystem with a predator having an advantageous hunting strategy over the other. Moreover, we consider interspecific competition between the two predator species. We have discussed the existence (including uniqueness), local and global stability of trivial equilibria, boundary equilibria (only one predator species present equilibria) and interior equilibria (both predator species coexistence equilibria). When death rate d_2 of first predator increases then first predator faces extinction and as a result $E_2 \rightarrow E_1$. Also, when predator advantage coefficient m of second predator decreases then it faces extinction and as a result $E_3 \rightarrow E_1$. From this we can conclude that hunting advantage (m) of the predator (p_2) has a positive effect on its survivability. Moreover, it helps in stabilizing p_2 's existence. As shown in the previous section, co-existence equilibrium point E^* is not stable which verifies the fact that our model holds the competitive exclusion principle firmly agreeing with the findings of Yuhua Long et. al. [22]. That is, one predator population anyhow outnumbers the other predator population when competing for the same prey. Also, for some parameter values it shows bistability which gives that the above discussed biosystem (2) depends heavily on the initial population of the species. For the same parameter values, some initial populations lead to the collapse of the first predator population (p_1) at the same time, some other initial population leads to the extinction of the second predator population (p_2). That is, depending on the initial populations of the two predator species, one predator species drives the extinction of the other predator species.

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