Original Article

An Analysis Of A Two Predator One Prey Model With Predator Advantage

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Abstract

In this study, a mathematical model of two predators one prey is developed. We take into account the direct competition between the two predator species for food .Furthermore, we assume that one of the predators has a tactical edge over the other. All biologically viable equilibrium points are calculated analytically and numerically. The main mathematical components of the model have been thoroughly investigated. Numerical simulations are carried out to support and illustrate the analytical conclusions.

Keywords: Predator–prey, Inter-specific competition,Predator advantage,Functional response,Stability

1 Introduction

Numerous researchers have expressed a strong interest in the population interaction model. One of the most thoroughly studied areas of ecology is the study of interactions between species. An important area of applied mathematics and theoretical ecology is the study of dynamical behaviour and interactions between prey and predator. Even though a lot of work has been done in this area, many ecological and modelling problems still need to be resolved. Therefore, Hence, the mathematical analysis of prey–predator systems requires the efforts of both mathematicians and ecologists.Different ecological models have been created, and they have greatly helped us understand how prey and predator interactions work. The dynamic behaviour of interacting species on an environment can be studied with the aid of ecological modelling [1]. A.J. Lotka and V. Volterra's works [2] [3] are where the study of the dynamics of predator-prey systems originated. The fundamental model is the Lotka-Volterra model, which uses a set of non-linear coupled first-order ordinary differential equations to describe the interaction of two species. Numerous extensions have been suggested to account for the various ecological phenomena. One of the modification that has been made is the addition of a new prey or predator. For instance, some researchers have created mathematical models that represent the interaction between one prey and two predators, and they have researched dynamics such the stability of equilibrium states to identify the conditions of coexistence in ecosystems, the extinction of species, the presence of stable periodic solutions, etc. [4–8]. There are also numerous mathematical models depicting how two preys interact with a single predator. Some Researchers [9] also studied models describing interaction between two predators and two prey species.

Competition describes the situation when two populations use the same limited resource and fight with each other for the resource to survive [5]. Resource competition is a well-known occurrence in both nature and society [7]. Intraspecific competition is between individuals of the same species, whereas interspecific competition is between individuals of different species.When competition between species is incorporated, the system with two predators and one prey becomes more sophisticated than the Lotka Voltera system. B.Mukhopadhyaya [4] studied a two predator and one prey model with competition among the predator species. B.Dubey et.al [10] proposed and analysed a model of two predators competing for a single prey. He showed that that the role of food conversion coefficients of predators in ratio-dependent models are crucial in determining the stability behavior of planer equilibria.Dian Savitri et.al [6] has studied the dynamics of a modified Leslie–Gower one prey–two predators model with competition between predator populations.The mathematical definition of predator-prey interaction, referred to as a functional response, is an important aspect of predator-prey population modelling. An essential component of the prey-predator relationship is the predator's rate of feeding on its prey, or what is known as the functional response of the predator. This rate is measured by the average number of prey killed by each predator per unit of time [1].There are several types of functional responses we can use .In 1959, Holling [11] explained some characteristics of the types of predation and gave three different types of functional responses and they are: 1.Holling Type-I functional response-

$f_1(x) = gx$

is characterized by a linear increase of intake rate with the amount of food available. Linear increase assumes that food processing or food searching time and other limitations are negligible. Animals just eat what they can get.

2.Holling Type-II functional response-

In 1913, Michaelis and Menten introduced a non-linear Holling type II response function to study the enzymatic reactions

$$f_2(x) = \frac{jx}{h+x}$$

f_2 denotes intake rate and x denotes resource density. j, h are positive and stands for maximal growth rate and half saturation constant of the species respectively

It is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food.

Another type of functional response known as sigmoidal functional response-

$$f_3(x) = \frac{ix^2}{h+lx+mx^2}$$

3.Holling Type-III functional response-

$$f_4(x) = \frac{ix^2}{h+mx^2}$$

which is the simplified form of $f_3(x)$.

Although there are more different kinds of functional response can be found. [12–14,23].In this model, we will study the dynamics of the predator-prey system using the Holling type-I functional response, which assumes a linear increase in intake rate with food density, either for all food densities or only for food densities up to a maximum, beyond which the intake rate is constant. The linear increase assumes that the time required by the consumer to process a food item is insignificant, or that consuming does not interfere with foraging for food. Dian Savitri et.al [5] has studied a modified Leslie–Gower one prey–two predators model with competition between predator populations using Holling type-I functional response for both the predators.Sahabuddin Sarwardi [15] studied a three-component model consisting on one-prey and two-predator populations is considered with a Holling type II response function incorporating a constant proportion of prey refuge. D. Didiharyono [16] discussed stability of one prey two predator with Holling type III and with harvesting at second predator population. Zhen Wang et.al [17] discussed a delayed generalized fractional-order prey-predator model with interspecific competition. Kalyan Manna et. al [18] considered a three species prey–predator model where the predator is generalist in nature as it survives on two prey species with intra-specific competition between the two prey species. Nijamuddin Ali et.al [19] studied a ratio-dependent food chain model where the total population has been divided into three classes, namely prey, predator and top-predator population. They have also incorporated intra-specific competition of predators in the model. Some predators devise strategies to gain an advantage over other predator species. Lions, piranhas, killer whales, and ants all have one thing in common: they have all acquired the tendency to hunt in groups. Group hunting is one of the most intriguing animal behaviours, with a huge variety of various behaviours that animals employ to acquire prey [24,25]. Social predation enables predator groups to locate, target, and kill bigger or more numerous prey with higher efficiency than single individual can achieve alone. Individual predators can gain various fitness advantages through social foraging, such as improved prey detection, the acquisition of more resources, and social or collective hunting, which is regarded to have played a significant role in the evolution of group life. For instance, bigger lion prides (Panthera leo) may take down bigger prey than smaller prides. [20]. Many creatures, like Barn Owls, Bats, Leopards, and many more, rely on their acute senses to survive and hunt in the faint light of the night sky. These nocturnal hunters use a mix of keen hearing and night vision to seek their prey under the cover of darkness while avoiding the competition of daylight. By avoiding the competition during daylight hours, this behaviour offered them an edge over other predators. Spotted hyenas can be active both day and night, depending on their needs .

After taking into account a few presumptions, we will explore a mathematical model of two predators vying for the same prey with one predator species employs a beneficial hunting strategy as compared to other .We introduce the aforesaid model in section 2.In section 3 , positivity and boundedness of the solution,local stability analysis are discussed.In section 4, Global stability analysis are done .In Section 5, numerical simulations to verify the findings are done followed by discussion and conclusion in Section 6.

2 Model formulation

Before formulating the model, we make the following assumptions:

1. We consider a model of Two predator species competing for one prey species where s is the population size of the prey species,$p_1$and p_2 are the population sizes of the predator species at any time t.

2. We assume that prey growth is logistic in nature in the absence of the predator, owing to limited resources in nature. Where the carrying capacity of the eco system is k.
3. Since there are two predator populations, they will eat prey, which may be termed as interaction of the prey with two predators and the mathematical term dealing with these interactions is called functional responses.There are various types of functional responses that we can use according to our model.Here we use Holling Type-1 functional response for both the predators.The terms sp_1r_2 and sp_2r_3 denote the first and second predators' response to the prey species, respectively.

4. Competition occurs when access to resources is negatively affected by the presence of other individuals.Predator competition can occur among conspecific species and/or heterospecific species sharing prey resources. Intraspecific competition decreases consumption rates within a single species, while interspecific competition decreases consumption rates of other species sharing the same resources. We assume There is a competition [4] among the two predators which may be described mathematically in general as $k_1p_1p_2$ and $k_2p_1p_2$.

5. We consider a natural death among the two predator species.

Under the above assumptions, we have the following model equations:

$$\begin{aligned} \frac{ds}{dt} &= r_1 s(1-\frac{s}{k}) - d_1 s - sp_1r_2 - sp_2r_3 \\ \frac{dp_1}{dt} &= r_4r_2sp_1 - k_1p_1p_2 - d_2p_1 \\ \frac{dp_2}{dt} &= r_5sp_2r_3 - k_2p_1p_2 - d_3p_2 \end{aligned} \tag{1}$$

which is studied by Hsu et.al. [21].The authors studied the relationship between the coefficient of interference and competition outcome.

Here, we use logistic growth for the prey in absence of the predator which is given by the term $r_1 s(1-\frac{s}{k})$.Here if s=k then this term becomes zero i,e at the saturation point ,there is no growth. We assume that the prey is very much smaller in size as compared to both the predator species and hence the handling time is negligible .So, Holling Type-I functional response is appropriate here.

The first predator(p_1) consumes the prey(s) with Holling Type-I functional response r_2sp_1 and contributes to its growth with rate $r_4r_2sp_1$.Similarly,The 2nd predator(p_2) consumes the prey(s) with Holling Type-I functional response r_3sp_2 and contributes to its growth with rate $r_5r_3sp_2$.

We assumes that one predator(p_2)has an advantageous hunting strategy (e.g.: night hunting,etc) compared to the other predator (p_1) which increases its chances of successful hunting.

The above model has been updated by incorporating a predator advantage in p_2 .Here m is the predator advantage coefficient which is proportional to prey density ms.

To incorporate the predator advantage effect, we multiply the term (1+m) in the functional response of the second predator i,e $r_3sp_2(1 + m)$

Under these additional effects, the above system 1 reduces to the following modified form:

$$\frac{ds}{dt} = r_1 s(1 - \frac{s}{k}) - sp_1r_2 - sp_2r_3(1+m)$$
$$\frac{dp_1}{dt} = r_4r_2sp_1 - k_1p_1p_2 - d_2p_1$$
$$\frac{dp_2}{dt} = r_5sp_2r_3(1+m) - k_2p_1p_2 - d_3p_2 \tag{2}$$

with initial conditions:

$$s(0) = s^0 > 0, p_1(0) = p_1^0 > 0, p_2(0) = p_2^0 > 0$$

In the above model, We consider:

3 Analysis of the Proposed Model

In this section, we study positivity and boundedness,existence of equilibrium points of the proposed system (2) and examine their stability.

System parameter	Significance of the parameter
s, p_1, p_2	Population sizes of the prey,first predator and second predator respectively at any time t,
r_1 and k	Intrinsic growth rate and environmental carrying capacity for the prey respectively
d_2, d_3	Death rate of the prey, first predator and the second predator respectively
k_1, k_2	The interference rate for the first predator and second predator respectively
r_2, r_3	Predation rate for the first predator and the second predator respectively
r_4, r_5	predator's conversion efficiency of the two predator species
m	predator advantage coefficient

Table 1: Significance of system parameters.

3.1 Positivity and Boundedness of the solution

:

It is important to show the positivity and the boundedness of the solution of the system (2) as they represent populations. To show that all the solutions of our system (2) are positive at first let us write our present system (2) as follows:

$$\frac{dS}{dt} = S\phi_1(S, P_1, P_2)$$
$$\frac{dP_1}{dt} = P_1\phi_2(S, P_1, P_2)$$
$$\frac{dP_2}{dt} = P_2\phi_3(S, P_1, P_2) \tag{3}$$

We now use the following lemma to show that all the solutions of our system are positive.

Lemma 1: Any solution of the differential equation $\frac{dX}{dt} = X\chi(X,Y)$ is always positive.

Theorem 1: Solutions to system (2) are always positive.

Proof: Since system (2) can be written in the form of system 3 then the proof of the theorem follows from Lemma 1.

Theorem 2: The solution of the system (2) exists in R_+^3 for t ≥ 0 and it is bounded.

Proof: System (1) is bounded given by [21].One can easily find that system (2) is also bounded by the same notion.

3.2 Equilibrium analysis

:

In this section, we discuss the equilibrium points of system (2) with their existence conditions.Mathematically, there exist six types of equilibrium points but biologically we have only five and they are as follows:

(a) Trivial equilibrium point, $E_0 \equiv (0,0,0)$ and it always exist. It is the scenario where no prey as well as no predator exists **The boundary points:**

(b) Predators free equilibrium point,$E_1 \equiv (k,0,0)$ and it always exists;

It is the scenario where prey attains carrying capacity and there will be no predators.

(c) Boundary equilibrium point which is free from predator P_2, $E_2 \equiv (B_1,C_1,0)$ Where,

$B_1 = \frac{d_2}{r_2 r_4}$,$C_1 = \frac{r_1(-d_2+kr_2r_4)}{kr_2^2r_4}$

and it exists under the condition$k > \frac{d_2}{r_2r_4}$;

It is the scenario which describes extinction ofP_2.

(d) Boundary equilibrium point which is free from predator P_1, $E_3 \equiv (B_2,0,D_1)$ Where,

$B_2 = \frac{d_3}{(1+m)r_3r_5}$,$D_1 = \frac{r_1(-d_3+kr_3r_5+kmr_3r_5)}{k(1+m)^2r_3^2r_5}$

and it exists under the condition; $k > \frac{d_3}{r_3r_5+mr_3r_5}$

It is the scenario which describes extinction of P_1.

(e) Prey free equilibrium point,$E_4 \equiv (0,C_2,D_2)$ Where,

$C_2 = -\frac{d_3}{k_2}$, $D_2 = -\frac{d_2}{k_1}$

and it doest not exist;

It is the scenario which describes extinction of prey species.But if prey species goes extinct then the system will collapse i,e it is not biologically feasible.

(f) Interior equilibrium point which represents the coexistence of the species,$E^* \equiv (s^*, p_1^*, p_2^*)$

Where,

$$s^* = -\frac{-kk_1k_2r_1-d_3kk_1r_2-d_2kk_2r_3-d_2kk_2mr_3}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

$$p_1^* = -\frac{d_3k_1r_1+d_3kr_2r_3r_4+d_3kmr_2r_3r_4-kk_1r_1r_3r_5-kk_1mr_1r_3r_5-d_2kr_3^2r_5-2d_2kmr_3^2r_5-d_2km^2r_3^2r_5}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

$$p_2^* = -\frac{d_2k_2r_1-kk_2r_1r_2r_4-d_3kr_2^2r_4+d_2kr_2r_3r_5+d_2kmr_2r_3r_5}{k_1k_2r_1+kk_2r_2r_3r_4+kk_2mr_2r_3r_4+kk_1r_2r_3r_5+kk_1mr_2r_3r_5}$$

3.3 Local stability analysis of equilibrium points

:

The local behavior of system (2) is investigated by considering its linear approximation around each equilibrium state. The variational matrix of the linearized system around a point (s,p_1,p_2) is given by

$$J(s,p_1,p_2) = \begin{bmatrix} -p_1r_2-(1+m)p_2r_3-\frac{r_1s}{k}+r_1(1-\frac{s}{k}) & -r_2s & -(1+m)r_3s \\ p_1r_2r_4 & -d_2-k_1p_2+r_2r4s & -k_1p_1 \\ (1+m)p_2r_3r_5 & -k_2p_2 & -d_3-k_2p_1+(1+m)r_3r_5s \end{bmatrix}$$

Now we have the following:

i) The variational matrix around the trivial equilibrium point E_0 is

$$J(E_0) = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}$$

The eigenvalues of the variational matrix $J(E_0)$ are r_1, $-d_2$ and $-d_3$.Here,one eigenvalue is positive and two are negative. Hence the predator prey system (2) is always unstable around trivial equilibrium point E_0.

ii) The variational matrix around the predator free equilibrium point E_1 is

$$J(E_1) = \begin{bmatrix} -r_1 & -kr_2 & -k(1+m)r_3 \\ 0 & -d_2 + kr_2r_4 & 0 \\ 0 & 0 & -d_3 + k(1+m)r_3r_5 \end{bmatrix}$$

The eigenvalues of $J(E_1)$ are $-r_1$, $-d_2 + kr_2r_4$, $-d_3 + k(1 + m)r_3r_5$. Now, if the conditions $k < \frac{d_2}{r_2r_4}$ and$k < \frac{d_3}{r_3r_5+r_3mr_5}$ holds , then all three eigenvalues are purely negative.Hence, equilibrium E_1 is locally asymptotically stable if the aforesaid two conditions hold simultaneously . iii) The variational matrix around the predator free equilibrium point E_2 can be calculated as

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where,

$$a_{11} = r_1(1 - \frac{d_2}{kr_2r_4}) - \frac{d_2r_1}{kr_2r_4} - \frac{r_1(-d_2 + kr_2r_4)}{kr_2r_4}$$

,

$$a_{12} = -\frac{d_2}{r_4}, a_{13} = -\frac{(d_2(1+m)r_3)}{r_2r_4},$$

$$a_{21} = \frac{r_1(-d_2 + kr_2r_4)}{kr2}, a_{22} = 0, a_{23} = -\frac{k_1r_1(-d_2 + kr_2r_4)}{kr_2^2r_4},$$

$$a_{31} = 0, a_{32} = 0, a_{33} = -d_3 - \frac{k_2r_1(-d_2 + kr_2r_4)}{kr_2^2r_4} + \frac{d_2(1+m)r_3r_5}{r_2r_4}$$

The characteristic equation of the above variational matrix is

$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0$,

where

$F1 = -(a11 + a33)$,

$F2 = a11a33 - a12a21$

$F3 = a12a21a33$

If we choose

$$k > \frac{d_2}{r_2 r_4}$$

and

$$k_2 < -\frac{d_2 k r_2 r_3 r_5}{d_2 r_1 - k r_1 r_2 r_4}$$

and

$$d_3 > \frac{d_2 k_2 r_1 - k k_2 r_1 r_2 r_4 + d_2 k r_2 r_3 r_5}{k r_2^2 r_4}$$

and

$$m < \frac{-d_2 k_2 r_1 + k k_2 r_1 r_2 r_4 + d_3 k r_2^2 r_4 - d_2 k r_2 r_3 r_5}{d_2 k r_2 r_3 r_5}$$

then $F_i > 0$ where i = 1, 2, 3 and $F_1 F_2 > F_3$. Hence it follows Routh-Hurtwitz criteria of local asymptotic stability. Thus the system (2) is locally asymptotically stable around positive equilibrium E_2. iv) The variational matrix around the predator free equilibrium point E_3 can be calculated as

$$J(E_3) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where,

$$b_{11} = r_1(1 - \frac{d_3}{k(1+m) r_3 r_5}) - \frac{(d_3 r_1)}{k(1+m) r_3 r_5} - \frac{r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m) r_3 r_5}, b_{12} = -\frac{(d_3 r_2)}{(1+m) r_3 r_5)}, b_{13} = -\frac{d_3}{r_5}$$

$$b_{21} = 0, b_{22} = -d_2 + \frac{d_3 r_2 r_4}{(1+m) r_3 r_5} - \frac{k_1 r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)^2 r_3^2 r_5}, b_{23} = 0,$$

$$b_{31} = \frac{r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m) r_3}, b_{32} = -\frac{(k_2 r_1(-d_3 + k r_3 r_5 + k m r_3 r_5)}{k(1+m)^2 r_3^2 r_5)}, b_{33} = 0$$

,

The characteristic equation of the above variational matrix is

$\Lambda^3 + H_1 \Lambda^2 + H_2 \Lambda + H_3 = 0$,

where

$H1 = -(b11 + b22)$,

$H2 = b11b22 - b13b31$

$H3 = b13b31b22$

If we choose

$$k > \frac{d_3}{r_3 r_5 + m r_3 r_5}$$

and

$$r_2 < \frac{-d_3 r_3 r_5 - d_3 m r_3 r_5 + k r_3^2 r_5^2 + 2 k m r_3^2 r_5^2 + k m^2 r_3^2 r_5^2}{d_3 r_4}$$

and $r1 < kr2r4$

and

$$k_1 > \frac{-d_3 k r_2 r_3 r_4 - d_3 k m r_2 r_3 r_4}{d_3 r_1 - k r_1 r_3 r_5 - k m r_1 r_3 r_5}$$

from the above variational matrix, then $H_i > 0$ where i = 1, 2, 3 and $H_1 H_2 > H_3$. Hence it follows Routh-Hurtwitz criteria of local asymptotic stability. Thus the system (2) is locally asymptotically stable around positive equilibrium E_3.

v)The variational matrix around the coexistence equilibrium point E^*can be calculated as

$$J(E^*) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where,

$$c_{11} = (1 - \frac{s^*}{k})r_1 - \frac{(s^* r1)}{k} - p_1^* r_2 - p_2^*(1+m)r_3, c_{12} = -s^* r_2, c_{13} = -s^*(1+m)r_3,$$

$$c_{21} = p_1^* r_2 r_4, c_{22} = -d_2 - p_2^* k_1 + s^* r_2 r_4, c_{23} = -p_1^* k_1,$$

$$c_{31} = p_2^*(1+m)r_3 r_5, c_{32} = -p_2^* k_2, c_{33} = -d_3 - p_1^* k_2 + s^*(1+m)$$

The characteristic equation of the above variational matrix is

$$\lambda^3 + G_1\lambda^2 + G_2\lambda + G_3 = 0 \tag{4}$$

where

G1 = −(c11 + c22 + c33),

G2 = c11c33 + c11c22 + c22c33 −c12c21 −c13c31 −c23c32

G3 = c11(c22c33 −c23c32) + c12(c23c31 −c21c33) + c13(c21c32 −c22c31)

By the Routh-Hurwitz criterion, $E^*$is locally asymptotically stable if and only if

$G_1 > 0, G_3 > 0$ and $G_1 G_2 - G_3 > 0$

Theorem:3 The local stability of the equilibrium points of system ((2)) is given by

(i)E_0 is always unstable.

(ii)

holds. E_1 is locally asymptotically stable if conditions $k < \frac{d_2}{r_2 r_4}$ and $k < \frac{d_3}{r_3 r_5 + r_3 m r_5}$ holds.

(iii) The system (2) around predator free equilibrium point E_2 is locally asymptotically stable if conditions

$$k > \frac{d_2}{r_2 r_4}$$

and

$$k_2 < -\frac{d_2 k r_2 r_3 r_5}{d_2 r_1 - k r_1 r_2 r_4}$$

$$d_3 > \frac{d_2k_2r_1 - kk_2r_1r_2r_4 + d_2kr_2r_3r_5}{kr_2^2r_4}$$

and

and

$$m < \frac{-d_2k_2r_1 + kk_2r_1r_2r_4 + d_3kr_2^2r_4 - d_2kr_2r_3r_5}{d_2kr_2r_3r_5}$$

holds.

(iv) E_3 is locally asymptotically stable if conditions

$$k > \frac{d_3}{r_3r_5 + mr_3r_5}$$

and

$$r_2 < \frac{-d_3r_3r_5 - d_3mr_3r_5 + kr_3^2r_5^2 + 2kmr_3^2r_5^2 + km^2r_3^2r_5^2}{d_3r_4}$$

and

$$r_1 < kr_{2r_4}$$

and

$$k_1 > \frac{-d_3kr_2r_3r_4 - d_3kmr_2r_3r_4}{d_3r_1 - kr_1r_3r_5 - kmr_1r_3r_5}$$

holds.

(v)The coexistence equilibrium state $E^*$is locally asymptotically stable if and only if G_1,G_3 and $G_1G_2 - G_3$ are positive.

4 Global Stability

4.1 Global Stability of E_1:

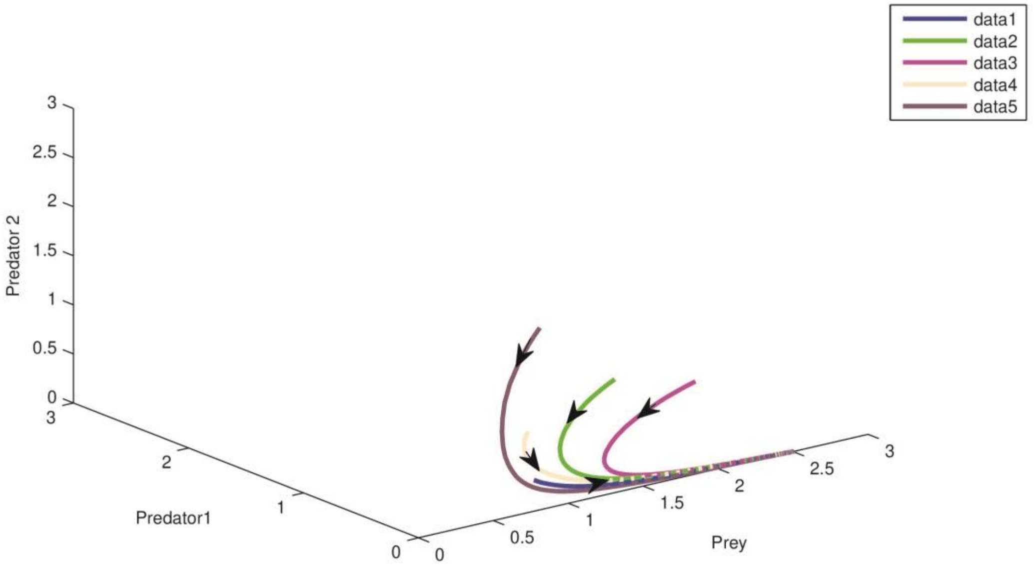

Figure 1: Axial Equilibrium point E_1=(k,0,0) is globally asymptotically stable.
Here,data1=(1,0.3,0.1),data2=(2,0.9,0.5),data3=(2,0.2,0.8),data4=(1.5,1,0.1),data5=(1.5,0.9,1.2)

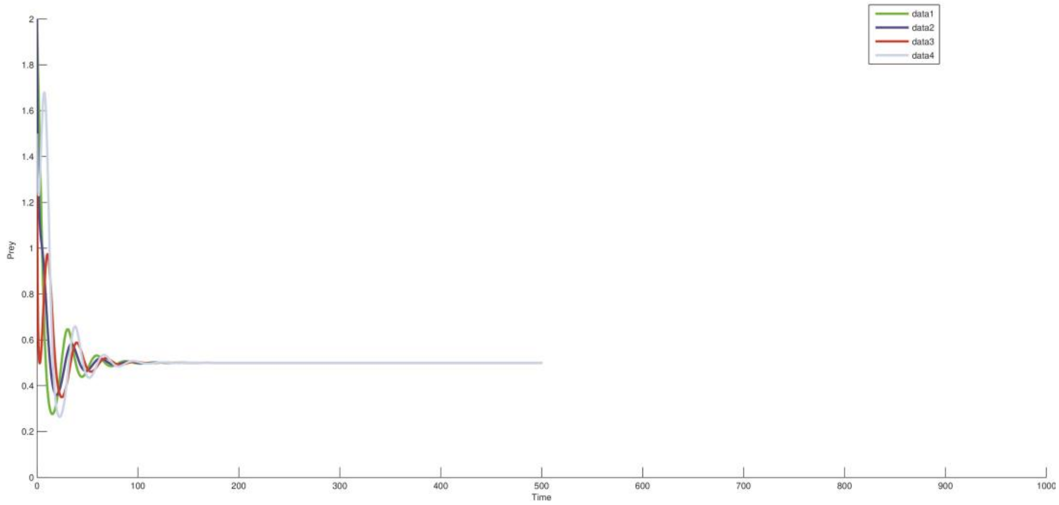

Figure 2: Time series of Prey around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, data1=(1.5,0.9,1.2),data2=(1.5,0.1,0.6),data3=(2,0.3,0.1),data4=(2,0.4,0.5)

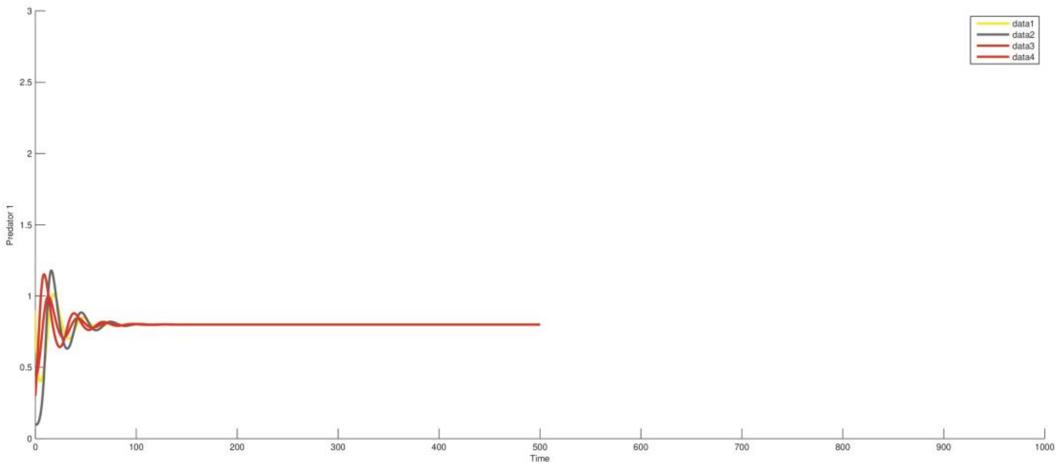

Figure 3: Time series of Predator 1 around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, data1=(1.5,0.9,1.2),data2=(1.5,0.1,0.6),data3=(2,0.3,0.1),data4=(2,0.4,0.5)

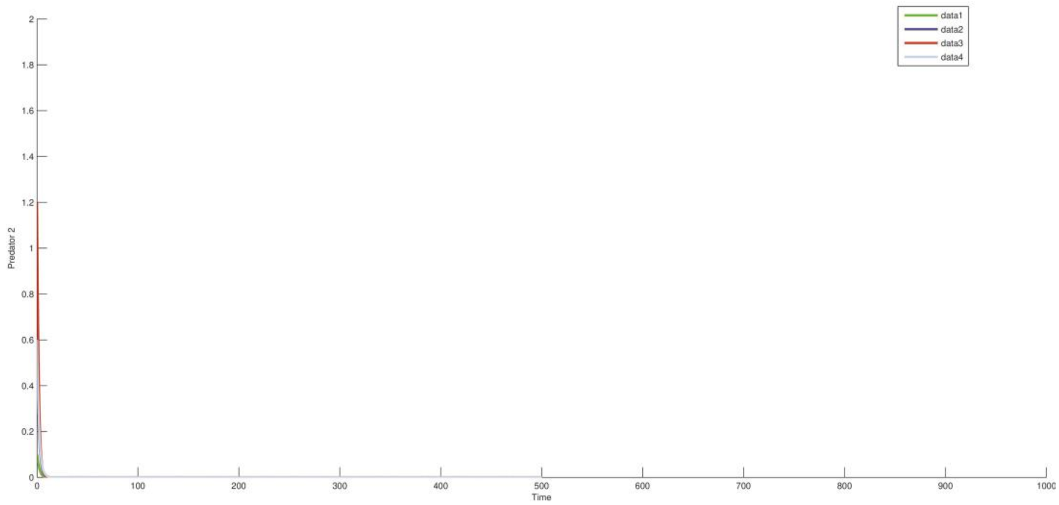

Figure 4: Time series of Predator 2 around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, data1=(1.5,0.9,1.2),data2=(1.5,0.1,0.6),data3=(2,0.3,0.1),data4=(2,0.4,0.5)

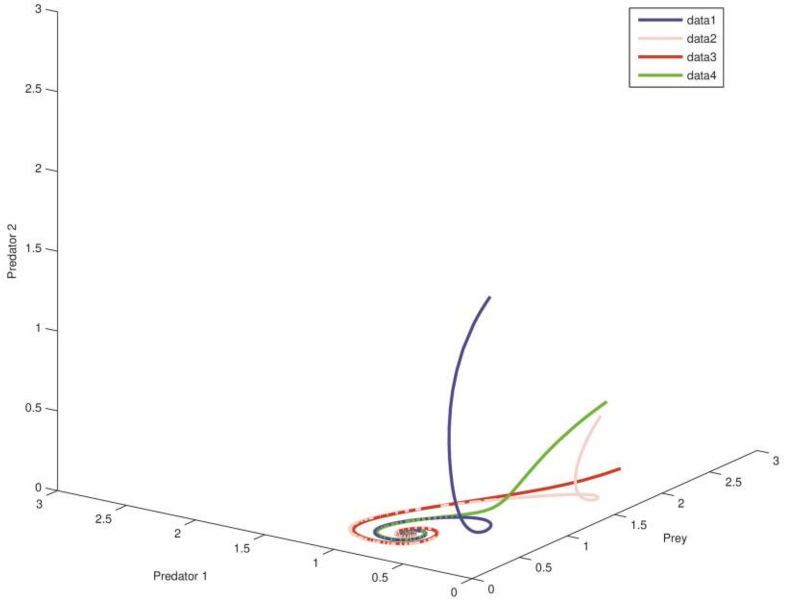

Figure 5: Phase portrait around E_2 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 0.5, d_2 = 0.125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.002, k = 2$. Here, data1=(1.5,0.9,1.2),data2=(1.5,0.1,0.6),data3=(2,0.3,0.1),data4=(2,0.4,0.5)

5 Numerical verification:

The main aim of this section is to verify the theoritical results developed in the previous sections with the help of MATHEMATICA software as well as MATLAB software. Real world data are not available for this model. Therefore, we use simulated data to illustrate the results obtained from our theoretical analysis.

Some hypothetical biotically feasible values of parameter have been considered as shown in the following table:

Parameter	value
r_1	0.5
r_2	0.5
r_3	0.5
r_4	0.5
r_5	0.25
d_3	0.7343
k_1	0.5
k	2.5

5.1 Equilibrium points:

To verify the global stabilty of the predators free equilibrium ,we take parameter values as follows: $m = 0.5$,$d_2 = 0.8125$,$k_2 = 0.5$ and the rest are from the above table.Many initial values of co-extant population have taken as shown in figure which converges to the axial equilibrium point which confirms the global stabilty and in turns confirms the local stabilty as shown in figure (1). However as the value of d_2 and k_2 decreses and when d_2=0.125,k_2=0.002 with all the other parameter values taking the same as above, E_1 loses its stabity and E_2 found to be existant as well as stable by Routh-Hurtwitz criteria as shown in figure (2,3,4,5) .On the other hand ,keeping the value of d_2 and k_2 fixed as $d_2 = 0.8125$,$k_2 = 0.5$ and then as the value of m increases m=3.8744,E_1 loses its stabity and E_3 found to be existant as well as stable by Routh-Hurtwitz criteria as shown in figure (6,7,8,9).Also, using MATHEMATICA software, we have found that for no combination of parameter values,the stability conditions of the co-existing equilibrium point E^*satisfies.Hence ,the result that $E^*$is unstable.

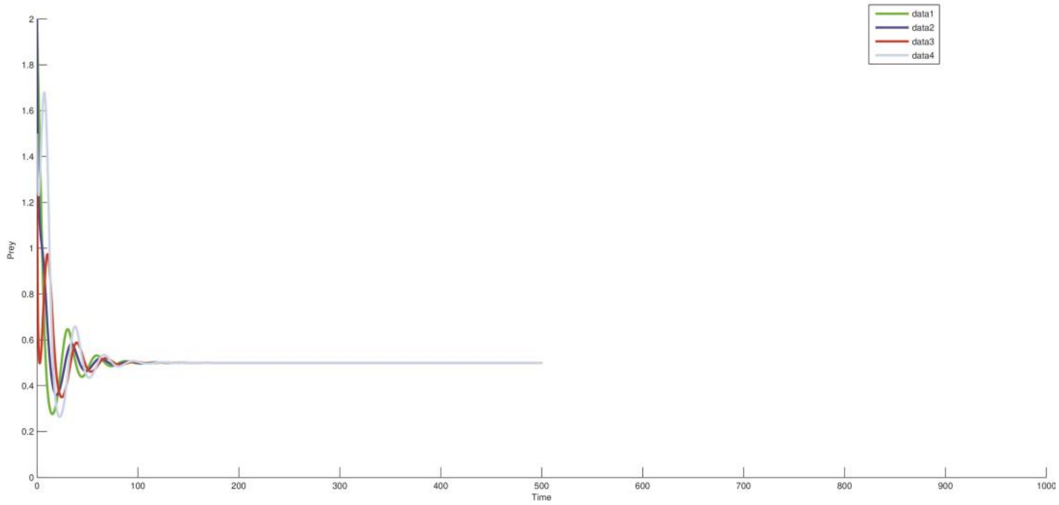

Figure 6: Time series of Prey around E_3 for parameter values $r_1 = 0.5$,$r_2 = 0.5$,$r_3 = 0.5$,$r_4 = 0.5$,$r_5 = 0.25$,$m = 3.8744$,$d_2 = 0.8125$,$d_3 = 0.7343$,$k_1 = 0.5$,$k_2 = 0.5$,$k = 2$. Here, data1=(2,0.3,0.5),data2=(2,0.1,0.8),data3=(1.5,0.5,1),data4=(1.5,0.9,0.8)

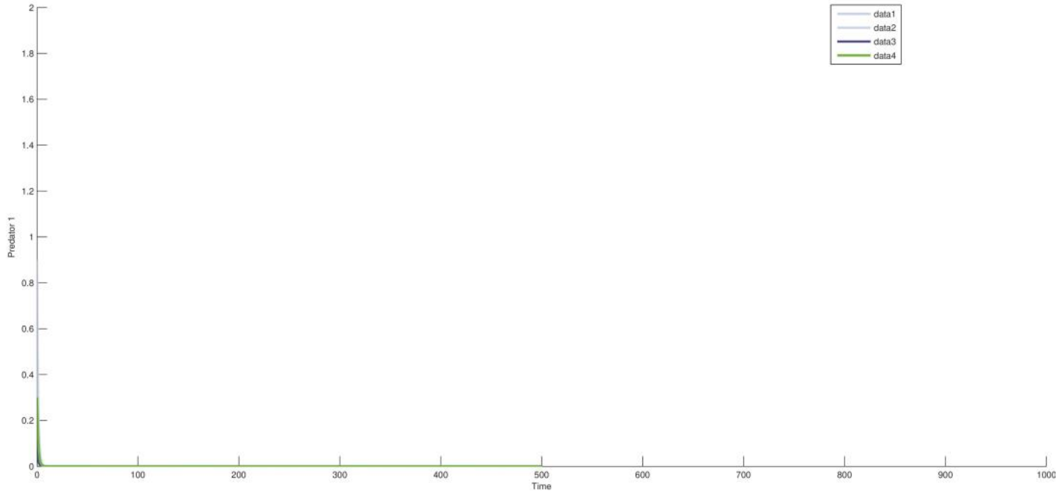

Figure 7: Time series of Predator 1 around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, data1=(2,0.3,0.5),data2=(2,0.1,0.8),data3=(1.5,0.5,1),data4=(1.5,0.9,0.8)

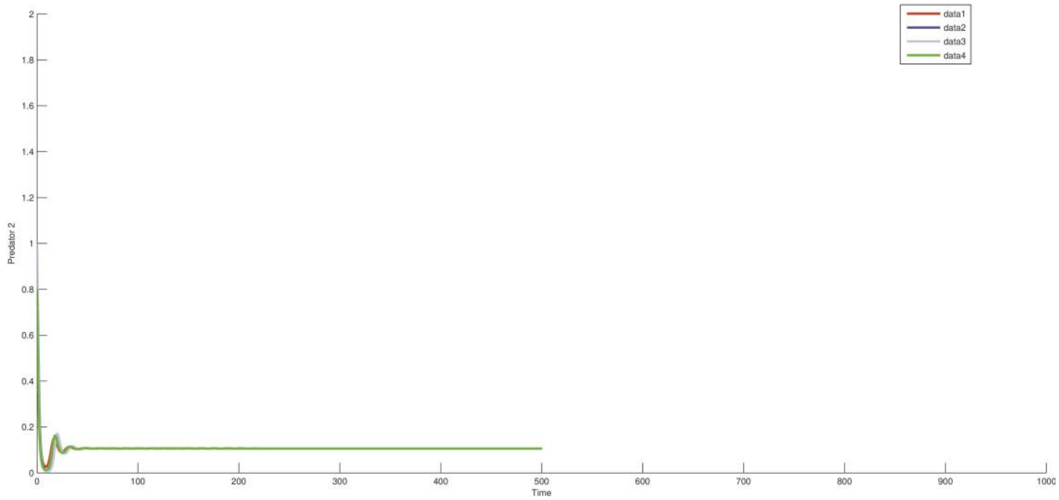

Figure 8: Time series of Predator 2 around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, data1=(2,0.3,0.5),data2=(2,0.1,0.8),data3=(1.5,0.5,1),data4=(1.5,0.9,0.8)

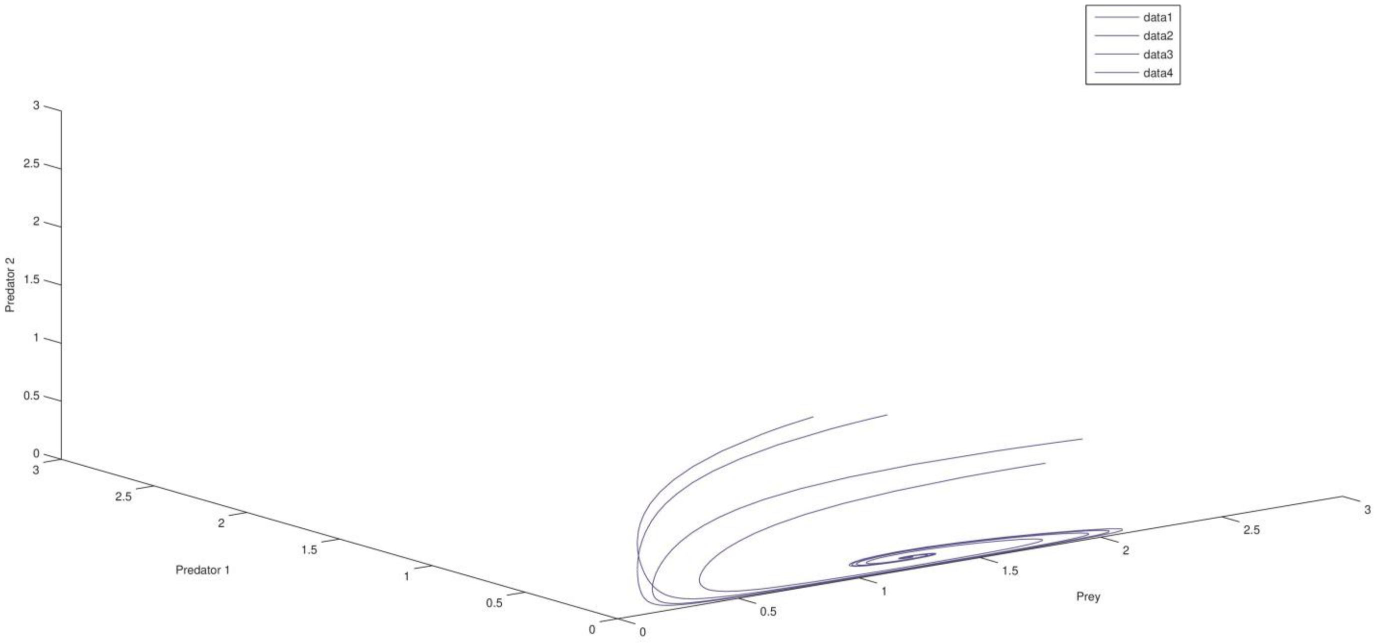

Figure 9: Phase portrait around E_3 for parameter values $r_1 = 0.5, r_2 = 0.5, r_3 = 0.5, r_4 = 0.5, r_5 = 0.25, m = 3.8744, d_2 = 0.8125, d_3 = 0.7343, k_1 = 0.5, k_2 = 0.5, k = 2$. Here, data1=(2,0.3,0.5),data2=(2,0.1,0.8),data3=(1.5,0.5,1),data4=(1.5,0.9,0.8)

5.2 Existence of Bistability :

Taking parameter values as $r_1 = 0.125, r_2 = 0.5, r_3 = 0.5, r_4 = 0.03125, r_5 = 0.0625, m = 0.25, d_2 = 0.015625, d_3 = 0.0625, k_1 = 0.5, k_2 = 0.5, k = 2$, it is found that it satisfies the Routh-Hurwitz criteria for E_2 as well as E_3.Thus,for these parameter values bothE_2 and E_3 found to be existent as well as stable .Hence bistability as shown in figure (10).

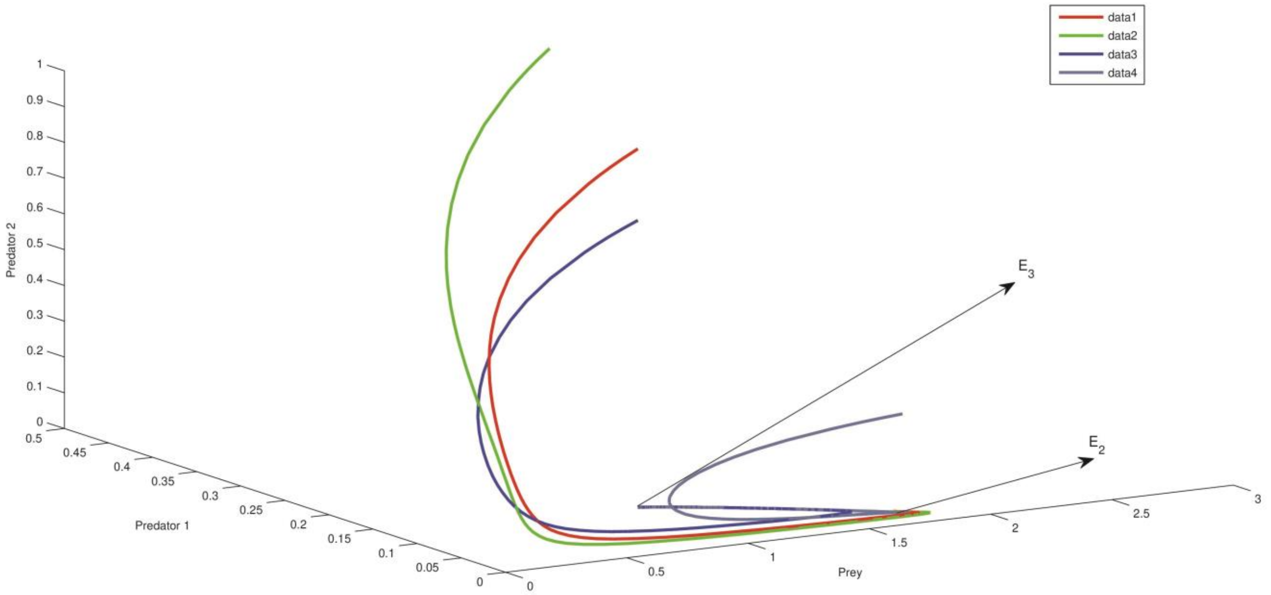

Figure 10: Existence of bistability showing existence of E_2 as well as E_3 simultaneously.
Here, data1=(2,0.4,0.7),data2=(2,0.5,0.9),data3=(2,0.4,0.5),data4=(2,0.1,0.2)

6 Discussion and Result:

In this paper,we consider a two predator one prey biosystem with a predator having an advantageous hunting strategy over the other.Morever we consider interspecific competition between the two predator species.We have discussed the existence (including uniqueness), local and global stability of trivial equilibria, boundary equilibria (only one predator species present equilibria) and interior equilibria (both predator species coexistence equilibria).When death rate d_2 of first predator increses then first predator faces extinction and as a result $E_2 \rightarrow- E_1$.Also, when predator advantage coefficient m for second predator decreses then it faces extinction and as a result $E_3 \rightarrow- E_1$.From this we can conclude that hunting advantage (m)of the predator(p_2) has a positive effect on its survivability.Morever, it helps in stabilizing p_2's existence. As shown in the previous section,co-existence equilibrium point $E^*$is not stable which verifies the fact that our model holds the competitive exclusion principle firmly agreeing with the findings of Yuhua Long et. al. [22].That is,one predator population anyhow outnumbers the other predator population when competing for the same prey.Also ,for some parameter values it shows bistability which gives that the above discussed biosystem (2) depends heavily on the initial population of the species .For the same parameter values, some initial populations leads to collapse of first predator population(p_1) at the same time,some other initial population leads to the extinction of the second predator population(p_2).That is,depending on the initial populations of the two predator species,one predator species drives extinction of the other predator species.

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