# Boundary Value Problem For A Fifth-Order Equation With Multiple Characteristics Containing The Second Time Derivative In A Finite Domain

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## I. Formulation of the problem.

In the region $D=\{(x,y):0<x,y<1\}$, consider the equation

$$\frac{\partial^5 u}{\partial x^5}+\frac{\partial^2 u}{\partial y^2}=0. \tag{1}$$

We will say that a $u(x,y)$ regular solution to equation (1) if it satisfies equation (1) in the domain $D$ and belongs to the class $C_{x,y}^{5,2}(D)\bigcap C_{x,y}^{4,1}(\bar{D})$ .

**Probem** $A$. Find a regular solution of equation (1) in a region $D$ that satisfies the boundary conditions

$$u_y(x,0)=0,\ u(x,1)=0, \tag{2}$$

$$\begin{cases} u(0,y)=\varphi_1(y),\ u_x(0,y)=\varphi_2(y), \\ u(1,y)=\varphi_3(y),\ u_x(1,y)=\varphi_4(y),\ u_{xx}(1,y)=\varphi_5(y), \end{cases} \tag{3}$$

where

$$\varphi_i(y)\in C^4[0,1],\ \varphi_i'(0)=\varphi_i'(1)=\varphi_i'''(0)=\varphi_i'''(1)=0, \qquad i=\overline{1,5}.$$

Note that in [1] an equation $\frac{\partial^{2n+1}}{\partial x^{2n+1}}u(x,y)+(-1)^n\frac{\partial^2}{\partial y^2}u(x,y)=f(x,y)$ with different boundary conditions was studied. In this work, when solving boundary value problems, the apparatus of potential theory was used. Also, in [2], a similar problem for a third-order equation was studied by the Fourier method. In [3,4] various boundary value problems in semi-infinite domains were studied for equation (1).

## II . Uniqueness of the solution

**Theorem-1.** If problem A has a solution, then it is unique.

**Proof.** Let problem A have two solutions $u_1(x,y)$ and $u_2(x,y)$. Then the function $u(x,y)=u_1(x,y)-u_2(x,y)$ satisfies equation (1) and homogeneous boundary conditions. Let us prove that $u(x,y)\equiv 0$ in $\overline{D}$. To do this, we multiply both parts of equation (1) by u , then we get

$$\frac{\partial}{\partial x}(uu_{xxxx})-\frac{\partial}{\partial x}(u_x u_{xxx})+\frac{1}{2}\frac{\partial}{\partial x}(u_{xx}^2)+\frac{\partial}{\partial y}(uu_y)-u_y^2=0,$$

integrating this identity over the domain $D$, we have

$$\int_0^1 u(1,y)u_{xxxx}(1,y)dy-\int_0^1 u(0,y)u_{xxxx}(0,y)dy-\int_0^1 u_x(1,y)u_{xxx}(1,y)dy+$$

$$+\int_0^1 u_x(0,y)u_{xxx}(0,y)dy+\frac{1}{2}\int_0^1 u_{xx}^2(1,y)dy-\frac{1}{2}\int_0^1 u_{xx}^2(0,y)dy+$$

$$+\int_0^1 u(x,1)u_y(x,1)dx-\int_0^1 u(x,0)u_y(x,0)dx-\iint_D u_y^2(x,y)dxdy=0.$$

Taking into account the homogeneous boundary conditions of the problem $A$, we obtain

$$\frac{1}{2}\int_0^1 u_{xx}^2(0,y)dy+\iint_D u_y^2(x,y)dxdy=0,$$

hence it follows that $u_y(x,y)=0$, then $u(x,y)=f(x)$. Given the condition $u(x,0)=0$, we get that $f(x)\equiv 0$ . Therefore $u(x,y)\equiv 0$, $(x,y)\in\overline{D}$. By virtue of the last equation, we obtain $u_1(x,y)=u_2(x,y)$. Theorem-1 is proved.

## III . Solution Existence

**Theorem-2.** If the functions $\varphi_i(y)\in C^4[0,1]$ and the matching conditions are met

$$\varphi_i'(0)=\varphi_i'(1)=0,\ \varphi_i'''(0)=\varphi_i'''(1)=0,\ i=\overline{1,5},$$

then the solution to the problem $A$ exists .

**Proof.** In order to prove the existence of a solution to the problem $A$, from the beginning we consider the following auxiliary problem: to find a non-trivial solution of equation (1) that satisfies the conditions and can be represented as

$$u(x,y)=X(x)Y(y). \qquad \text{(four)}$$

Putting (4) into (1), we get

$$X^{(5)}-\lambda^5 X=0, \qquad (5)$$

$$Y''+\lambda^5 Y=0, \qquad (6)$$

from (6) and (2) we will have

$$\begin{cases} Y'' + \lambda^5 Y = 0, \\ Y'(0) = 0,\ Y(1) = 0. \end{cases} \qquad (7)$$

The solution to problem (7) has the following form

$$Y(y) = C_1 \cos\sqrt{\lambda^5}\, y + C_2 \sin\sqrt{\lambda^5}\, y,$$

Taking into account the boundary conditions, we find nontrivial solutions of problem (7), exists for $\lambda > 0$, and these eigenvalues are [5] $\lambda_n = \left(\frac{\pi(2n-1)}{2}\right)^{\frac{2}{5}}$, $n = 1, 2, 3, \ldots$, and the eigenfunctions are

$$Y_n(y) = \sqrt{2}\cos\left(\frac{\pi(2n-1)}{2}\right) y. \qquad (8)$$

The general solution of equation (5) has

$$X(x) = C_1 e^{\lambda x} + e^{\lambda\alpha_2 x}\left(C_2 \cos\lambda\beta_2 x + C_3 \sin\lambda\beta_2 x\right) + e^{-\lambda\alpha_1 x}\left(C_4 \cos\lambda\beta_1 x + C_5 \sin\lambda\beta_1 x\right) \quad (9)$$

where

$$\alpha_1 = \cos\theta_1,\ \beta_1 = \sin\theta_1,\ \alpha_2 = \cos\theta_2,\ \ \beta_2 = \sin\theta_2,\ \theta_1 = \frac{\pi}{5},\ \theta_2 = \frac{2\pi}{5}.$$

$C_i - \left(i = \overline{1,5}\right)$ are arbitrary constants.

Due to the linearity and homogeneity of equation (1), any finite sum of solutions will also be solutions. Taking this into account, $A$ we are looking for a solution to the problem in

$$u(x, y) = \sqrt{2}\sum_{n=1}^{\infty}\left[C_1 e^{\lambda x} + e^{\lambda\alpha_2 x}\left(C_2 \cos\lambda\,\beta_2 x + C_3 \sin\lambda\,\beta_2 x\right) + \right.$$

$$\left. + e^{-\lambda\alpha_1 x}\left(C_4 \cos\lambda\,\beta_1 x + C_5 \sin\lambda\beta_1 x\right)\right]\cos\left(\frac{\pi(2n-1)}{2}\right) y. \quad (10)$$

The function defined by series (10) satisfies conditions (2), since all members of the series satisfy them. Satisfying conditions (3), we obtain the system of equations

$$\begin{cases} C_{1n}+C_{2n}+C_{4n}=A_{1n}, \\ C_{1n}+\cos\theta_2 C_{2n}+\sin\theta_2 C_{3n}-\cos\theta_1 C_{4n}+\sin\theta_1 C_{5n}=\dfrac{A_{2n}}{\lambda_n}, \\ C_{1n}e^{\lambda_n}+C_{2n}e^{\lambda_n\alpha_2}\cos\lambda_n\beta_2+C_{3n}e^{\lambda_n\alpha_2}\sin\lambda_n\beta_2+ \\ +C_{4n}e^{-\lambda_n\alpha_1}\cos\lambda_n\beta_1+C_{5n}e^{-\lambda_n\alpha_1}\sin\lambda_n\beta_1=A_{3n}, \\ C_{1n}e^{\lambda_n}+C_{2n}e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+\theta_2\right)+C_{3n}e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+\theta_2\right)- \\ -C_{4n}e^{-\lambda_n\alpha_1}\cos\left(\lambda_n\beta_1-\theta_1\right)-C_{5n}e^{-\lambda_n\alpha_1}\sin\left(\lambda_n\beta_1-\theta_1\right)=\dfrac{A_{4n}}{\lambda_n}, \\ C_{1n}e^{\lambda_n}+C_{2n}e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+2\theta_2\right)+C_{3n}e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+2\theta_2\right)+ \\ +C_{4n}e^{-\lambda_n\alpha_1}\cos\left(\lambda_n\beta_1-2\theta_1\right)+C_{5n}e^{-\lambda_n\alpha_1}\sin\left(\lambda_n\beta_1-2\theta_1\right)=\dfrac{A_{5n}}{\lambda_n^{\;2}}. \end{cases} \tag{11}$$

Where

$$A_{in}=\sqrt{2}\int_0^1\varphi_i(y)\cos\left(\frac{\pi(2n-1)}{2}\right)y\,dy,\ i=\overline{1,5}. \tag{12}$$

Solving system (11) we get

$$C_{in}=\frac{\Delta_i}{\Delta},\quad i=\overline{1,5}.$$

Let's show that $\Delta\neq 0$. To do this, we prove the following lemma.

**Lemma:** Boundary value problem

$$\begin{cases} X^{(5)}-\lambda^5 X=0, \\ X(0)=X'(0)=X(1)=X'(1)=X''(1)=0, \end{cases}$$

has only a trivial solution.

**Proof:** Assume the opposite, let $X(x)\neq 0$. Consider the identity

$$X\left(X^{(5)}-\lambda^5 X\right)=0,$$

or

$$\left(XX^{(4)}-X'X'''+\frac{1}{2}\left(X''\right)^2\right)'-\lambda^5X^2=0,$$

integrating over the region $(0<x<1)$, we have

$$\int_0^1\left(XX^{(4)}-X'X'''+\frac{1}{2}\left(X''\right)^2\right)'dx-\lambda^5\int_0^1X^2dx=0,$$

$$\left(XX^{(4)}-X'X'''+\frac{1}{2}\left(X''\right)^2\right)\Bigg|_0^1-\lambda^5\int_0^1X^2dx=0,$$

$$X(1)X^{(4)}(1)-X(0)X^{(4)}(0)-X'(1)X'''(1)+X'(0)X'''(0)+\frac{1}{2}\left(X''(1)\right)^2-$$

$$-\frac{1}{2}\left(X''(0)\right)^2-\lambda^5\int_0^1 X^2dx=0,$$

taking into account the boundary conditions, we obtain

$$\frac{1}{2}\left(X''(0)\right)^2+\lambda^5\int_0^1 X^2dx=0\,,$$

since $\lambda>0$, then $X(x)\equiv 0$.

Hence, the system of equations (11) has a unique solution. The determinant of the system has the form:

$$\Delta=\begin{vmatrix} A_{2\times3} & B_{2\times2} \\ C_{3\times3} & D_{3\times2} \end{vmatrix},$$

where

$$A_{2\times3}=\begin{vmatrix} 1 & 1 & 0 \\ 1 & \cos\theta_2 & \sin\theta_2 \end{vmatrix},\quad B_{2\times2}=\begin{vmatrix} 1 & 0 \\ -\cos\theta_1 & \sin\theta_1 \end{vmatrix},$$

$$C_{3\times3}=\begin{vmatrix} e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\lambda_n\beta_2 & e^{\lambda_n\alpha_2}\sin\lambda_n\beta_2 \\ e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+\theta_2\right) & e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+\theta_2\right) \\ e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+2\theta_2\right) & e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+2\theta_2\right) \end{vmatrix},$$

$$D_{3\times2}=\begin{vmatrix} e^{-\lambda_n\alpha_1}\cos\lambda_n\beta_1 & e^{-\lambda_n\alpha_1}\sin\lambda_n\beta_1 \\ -e^{-\lambda_n\alpha_1}\cos\left(\lambda_n\beta_1-\theta_1\right) & -e^{-\lambda_n\alpha_1}\sin\left(\lambda_n\beta_1-\theta_1\right) \\ e^{-\lambda_n\alpha_1}\cos\left(\lambda_n\beta_1-2\theta_1\right) & e^{-\lambda_n\alpha_1}\sin\left(\lambda_n\beta_1-2\theta_1\right) \end{vmatrix}.$$

Find the largest degree of the exponent included in the calculation of the determinant $\Delta$. Since $C_{3\times3}$ all the exponents in the determinant have positive degrees, it is obvious that the largest degree of the exponent is obtained by calculating the product of the following determinants

$$\left|C_{3\times3}\right|\cdot\left|B_{2\times2}\right|.$$

We calculate each determinant separately.

$$B=\begin{vmatrix} 1 & 0 \\ -\cos\theta_1 & \sin\theta_1 \end{vmatrix}=\sin\theta_1,$$

$$C=\begin{vmatrix} e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\lambda_n\beta_2 & e^{\lambda_n\alpha_2}\sin\lambda_n\beta_2 \\ e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+\theta_2\right) & e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+\theta_2\right) \\ e^{\lambda_n} & e^{\lambda_n\alpha_2}\cos\left(\lambda_n\beta_2+2\theta_2\right) & e^{\lambda_n\alpha_2}\sin\left(\lambda_n\beta_2+2\theta_2\right) \end{vmatrix}=4e^{\lambda_n+2\lambda_n\alpha_2}\sin\theta_2\sin^2\frac{\theta_2}{2}.$$

From here

$$\Delta = e^{\lambda_n+2\lambda_n\alpha_2}K + f\left(\lambda_n\right),$$

where

$$K = 4\sin\theta_1\sin\theta_2\sin^2\frac{\theta_2}{2},$$

$f\left(\lambda_n\right)=o\left(e^{\lambda_n+2\lambda_n\alpha_2}\right)$ at $\lambda_n\to+\infty$.

Let's estimate $\Delta$

$$\left|\Delta\right| = e^{\lambda_n+2\lambda_n\alpha_2}\left|K+e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)\right|,$$

because

$$\lim_{\lambda_n\to\infty} e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)=0,$$

then

$$\forall\varepsilon>0,\ \varepsilon<K\quad \exists N_1\big|\ \ \forall n>N_1\Rightarrow\left|e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)\right|<\varepsilon,$$

hence, for $n>N_1$, the inequality

$$\left|K+e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)\right|>K-\left|e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)\right|>K-\varepsilon,$$

denote

$$M_1=\min_{n=\overline{1,N_1}}\left|K+e^{-\left(\lambda_n+2\lambda_n\alpha_2\right)}f\left(\lambda_n\right)\right|,$$

according to the lemma $M_1\neq 0$, hence

$$\frac{1}{\left|\Delta\right|}\le\frac{1}{Me^{\lambda_n+2\lambda_n\alpha_2}},$$

where

$$M=\min\left\{M_1;K-\varepsilon\right\}.$$

Now we get estimates for $C_{in},\ \ i=\overline{1,5}$. Calculations show that the following estimates are valid for algebraic additions $\left|\Delta_i\right|,\ \ i=\overline{1,5}$:

$$\left|\Delta_1\right|\le M_1e^{2\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|.\ \left|\Delta_2\right|\le M_2e^{\lambda_n+\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|.\ \left|\Delta_3\right|\le M_3e^{\lambda_n+\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|.$$

$$\left|\Delta_4\right|\le M_4e^{\lambda_n+2\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|.\ \left|\Delta_5\right|\le M_5e^{\lambda_n+2\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|.$$

where

$$\left|A_{in}\right|\le\frac{2}{\left(\pi n\right)^4}\int_0^1\left|\varphi_i^{(4)}\right|dy\le\frac{S_i}{n^4},\quad M_i-const,\qquad i=\overline{1,5}.$$

Hence, for the coefficients $C_{in}$ we obtain the following estimates

$$\left|C_{1n}\right|=\frac{\left|\Delta_1\right|}{\left|\Delta\right|}\le\frac{M_1e^{2\lambda_n\alpha_2}\sum_{i=1}^{5}\left|A_{in}\right|}{Me^{\lambda_n+2\lambda_n\alpha_2}}\le\frac{N_1}{e^{\lambda_n}n^4},$$

$$\left|C_{2n}\right|=\frac{\left|\Delta_2\right|}{\left|\Delta\right|}\le\frac{M_2e^{\lambda_n+\lambda_n\alpha_2}\sum\limits_{i=1}^{5}\left|A_{in}\right|}{Me^{\lambda_n+2\lambda_n\alpha_2}}\le\frac{N_2}{e^{\lambda_n\alpha_2}n^4},$$

$$\left|C_{3n}\right|=\frac{\left|\Delta_3\right|}{\left|\Delta\right|}\le\frac{M_3e^{\lambda_n+\lambda_n\alpha_2}\sum\limits_{i=1}^{5}\left|A_{in}\right|}{Me^{\lambda_n+2\lambda_n\alpha_2}}\le\frac{N_3}{e^{\lambda_n\alpha_2}n^4},$$

$$\left|C_{4n}\right|=\frac{\left|\Delta_4\right|}{\left|\Delta\right|}\le\frac{M_4e^{\lambda_n+2\lambda_n\alpha_2}\sum\limits_{i=1}^{5}\left|A_{in}\right|}{Me^{\lambda_n+2\lambda_n\alpha_2}}\le\frac{N_4}{n^4},$$

$$\left|C_{5n}\right|=\frac{\left|\Delta_5\right|}{\left|\Delta\right|}\le\frac{M_5e^{\lambda_n+2\lambda_n\alpha_2}\sum\limits_{i=1}^{5}\left|A_{in}\right|}{Me^{\lambda_n+2\lambda_n\alpha_2}}\le\frac{N_4}{n^4}.$$

Let us prove the uniform convergence of series (10) in the region $D$.

$$\left|u(x,y)\right|\le\sum_{n=1}^{\infty}\left[\left|C_{1n}\right|\left|e^{\lambda_n x}\right|+\left(\left|C_{2n}\right|+\left|C_{3n}\right|\right)\left|e^{\lambda_n\alpha_2 x}\right|+\left(\left|C_{4n}\right|+\left|C_{5n}\right|\right)\left|e^{-\lambda_n\alpha_1 x}\right|\right]\le$$

$$\le\sum_{n=1}^{\infty}\left[\frac{N_1}{e^{\lambda_n(1-x)}n^4}+\frac{N_2}{e^{\lambda_n\alpha_2(1-x)}n^4}+\frac{N_3}{e^{\lambda_n\alpha_2(1-x)}n^4}+\frac{N_4}{n^4}+\frac{N_5}{n^4}\right]\le\sum_{n=1}^{\infty}\frac{N}{n^4}<\infty\ .$$

Similarly, the uniform convergence of a series composed of partial derivatives with respect to a variable $x$ up to the fifth order inclusive is shown.

For $u_{yy}(x,y)$ we have the estimate

$$\left|\frac{\partial^2u}{\partial y^2}\right|\le\left|\sum_{n=1}^{\infty}(\pi n)^2X_n(x)Y_n(y)\right|\le\sum_{n=1}^{\infty}(\pi n)^2\frac{N}{n^4}\le\sum_{n=1}^{\infty}\left(\frac{\pi}{n}\right)^2N<\infty.$$

Similarly

$$\left|\frac{\partial^5u}{\partial x^5}\right|\le\left|\sum_{n=1}^{\infty}X_n^{(5)}(x)Y_n(y)\right|\le\sum_{n=1}^{\infty}\left|X_n^{(5)}(x)\right|\left|Y_n(y)\right|=$$

$$=\sum_{n=1}^{\infty}\lambda_n^5\left|X_n(x)\right|\left|Y_n(y)\right|\le\sum_{n=1}^{\infty}(\pi n)^2\frac{N}{n^4}<\sum_{n=1}^{\infty}\left(\frac{\pi}{n}\right)^2N<\infty.$$

Theorem -2 is proven

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