

The Strongly Rainbow Connected Number of the Inverse Graphs

M. Lakshmi Kameswari¹, N. Naga MaruthiKumari², T.V. Pradeep Kumar³

¹Department of Mathematics, College of Engineering and Technology, Nagarjuna University, Guntur-522510, Andhra Pradesh.

²Department of Mathematics, Higher college of Technology, University of Applied Sciences, Muscat, Oman

³Department of Mathematics, College of Engineering and Technology, Nagarjuna University, Guntur-522510, Andhra Pradesh.

Email: lakshmikameswari.pavan@gmail.com

DOI: 10.47750/pnr.2022.13.S06.239

Abstract

The two key parameters in this study of graph theory are the rainbow connected number and strongly rainbow connected numbers. The purpose of this study is to identify the strongly rainbow connected numbers of the inverse graphs corresponding to the modular groups under various operations, and to evaluate and explain the characteristics of these numbers in various contexts.

The objective of this Study is to apply these concepts and results in the field of Networking and Coding. An edge colored graph G is called rainbow-connected if a path whose edges have different colors that connects any two vertices. The minimum k for which there exist a rainbow k -coloring of G is called the rainbow connection number of G , denoted by $rc(G)$, the adjacent edges may be allowed to color with the same color. The graph G is strongly rainbow connected if there exists a rainbow u - v geodesic for every pair of vertices u and v in G .

Keywords: Cyclic group, Self-invertible elements, Inverse graph, Geodesic, Rainbow connected number, strongly rainbow connected number and Tree.

INTRODUCTION

Studying the algebraic structures of groups based on graphs with Number Theory properties has created the interests of many researchers in recent Research areas for a few decades. Recently, graphs associated with groups has become more interesting and active area of Research.

This work incline more light on the algebraic structures of groups via graphs and explore for more researches in this approach. Many of the properties of graph and group theories are analogous for more than a century. Many investigations and surprising results from group theory to Graph Theory have been proved easily[10]via Number Theory properties. This creates the motivation for this Research.

Rainbow connected path:

A graph $GS(\tau)$ is said to be rainbow-connected if there is a rainbow path between each pair of its vertices. Let $GS(\tau)$ be a nontrivial connected graph on which is defined a coloring

$c : E(G) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$ of the edges of G , where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it.[3]

Geodesic: The pair of vertices u, v is said to be u - v geodesic, if the distance between the vertices u and v is equal to the length of the rainbow path between u and v . [3]

Rainbow connected number: Let $GS(\tau)$ be a nontrivial connected graph with an edge coloring: $E(G) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. An edge-colored graph G is called rainbow-connected if a path whose edges have different colors connects any two vertices. The minimum k for which there exist a rainbow k -coloring of G is called the rainbow connection number of G , denoted by $rc(G)$ [5]

Strongly rainbow connected number:

For every pair of vertices u and v of $GS(\tau)$ a rainbow u - v geodesic $GS(\tau)$ is a rainbow u - v path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v . The graph G is strongly rainbow connected if there exists a rainbow u - v geodesic for every pair of vertices u and v in G . The strongly rainbow connection number of G , denoted by $src(G)$, is the minimum number of colors that are needed in order to make G strongly rainbow connected. [5]

5. Inverse graph:

Let (τ, \circ) be a cyclic group and S be a non-empty subset of τ which consists of all non-self-invertible elements of τ with respect to the binary operation \circ . The inverse graph associated with the group τ denoted by $G_S(\tau)$, whose vertices are the elements of the group τ and the edges are formed by joining two distinct elements x, y of τ if and only if $x \circ y \in S$ or $y \circ x \in S$ [7]

Properties:

Lemma: In any tree $G_S(\tau)$ with m edges $rc(G_S(\tau)) = m = src(G_S(\tau))$

Proof: Let G be a tree with n vertices and m edges

As a tree consists of $(n-1)$ edges $m = n - 1$

$\forall u, v \in G_S(\tau), u, v$ is geodesic

Every pair of vertices in a tree is always geodesic

Since there exists a unique path between every pair of vertices in a tree that path

itself is a rainbow connection path

$\therefore d(u, v) =$ Minimum number of colors used in that path

$=$ length of the rainbow connection path

$\therefore (u, v)$ is geodesic $\forall u, v \in G_S(\tau)$

$\Rightarrow G_S(\tau)$ is strongly rainbow connected

$\therefore rc(G_S(\tau)) =$ The number of edges in the rainbow connection path

$=$ Number of edges in $G_S(\tau) = m = n - 1$

$= src(G_S(\tau))$

Theorem: The Inverse graph $G_S(Z_n)$ corresponding to the cyclic group $(Z_n, +)$ is Rainbow connected

and hence is Strongly Rainbow connected such that $rc(G_S(Z_n)) = src(G_S(Z_n)) = 2$

Proof: Let $(Z_n, +)$ is a cyclic group of order n

Let S be a non-empty subset of non-self-invertible elements of Z_n and $G_S(Z_n)$, be the corresponding inverse graph of Z_n . To investigate the Rainbow Connected Number, the following cases arise regarding the adjacency between every pair of vertices in $G_S(Z_n)$

Case 1 : Adjacency between the identity element and every element of S

The identity element '0' in $(Z_n, +)$ is adjacent to all the elements of S and

$$d(0, v) = 1 \forall v \in S$$

Case 2 : Adjacency between the elements of S

There arise 2 sub cases

Subcase 1: For any $x, y \in S$, if $x+y \in S \Rightarrow x, y$ are adjacent

Subcase 2: For any $x, y \in S$, if $x+y \notin S$, there arise the following observations.

If $x+y=0$, for $x \neq y$

Then in the inverse graph $G_S(Z_n)$, x, y are non-adjacent as $x^{(-1)}=y$

\exists an element $z \in Z_n - S$ ($z=0$ or $n/2$) such that $x+z \in S$ and $y+z \in S$

$\Rightarrow x, z$ are adjacent in $G_S(Z_n)$ as well as z, y are adjacent in $G_S(Z_n)$

$\Rightarrow \exists$ a path between x and y namely $x \rightarrow z \rightarrow y$

$d(x, y) =$ Minimum length of rainbow connected path between x and y is equal to 2

Hence there exists a rainbow connected path between every pair of elements

x, y in S , for $x+y=0$

$x+y=n/2$, x is neither the inverse of y nor adjacent to y (if n is even)

$\forall x \in Z_n \setminus \{0\}, \exists y (\neq x) \in Z_n$ such that $x+y=n/2$ and $x \neq y^{(-1)}$

\Rightarrow In the inverse graph $G_S(Z_n)$, x and y are non-adjacent

\exists an element $z \in Z_n - S$ ($z=0$ or $n/2$) such that $x+z \in S$ and $y+z \in S$

$\Rightarrow x, z$ are adjacent in $G_S(Z_n)$ as well as z, y are adjacent in $G_S(Z_n)$

$\Rightarrow \exists$ a path between x and y namely $x \rightarrow z \rightarrow y$

$d(x, y) =$ Minimum length of rainbow connected path between x and y is equal to 2

Hence there exists a rainbow connected path between every pair of elements x, y in S , for $x+y=n/2$

Case 3: Adjacency between two self-invertible elements when 'n' is even

\exists exactly two self-invertible elements $0, n/2 \in \mathbb{Z}_n - S$

Then there does not exist adjacency between 0 and $n/2$

\therefore There must exist an element say $z \in S$ such that \exists a path $0 \rightarrow z \rightarrow n/2$

$\therefore d(0, n/2) = 2$

Hence there exists a rainbow connected path between every pair of elements

$x, y \in \mathbb{Z}_n - S$

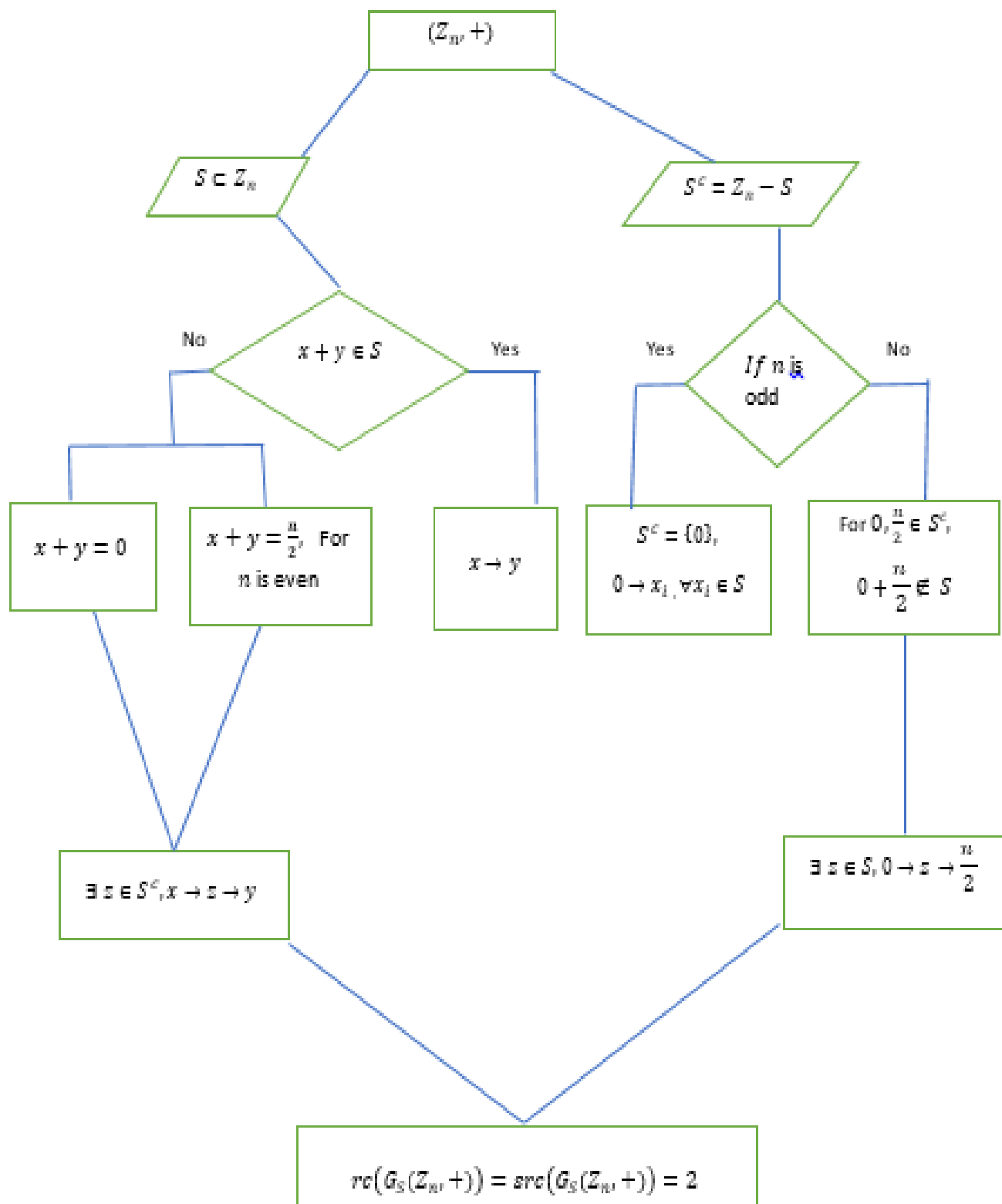
By considering all the 3 cases, $G_S(\mathbb{Z}_n)$ is Rainbow connected

x, y are geodesic and $G_S(\mathbb{Z}_n)$ is strongly rainbow connected $\forall x, y \in \mathbb{Z}_n$

$\therefore rc(G_S(\mathbb{Z}_n)) = src(G_S(\mathbb{Z}_n)) = 2$

Remark: If 'n' is even in $G_S(\mathbb{Z}_n)$, the self-invertible element $n/2$ is obviously adjacent to every element of S .

The following algorithm illustrates Rainbow connection path of the inverse graph $(G_S(\mathbb{Z}_n), +)$



Let us illustrate the above theorem by the following example.

In the Inverse graph $G_S(Z_7)$ corresponding to the cyclic group $(Z_7,+)$ is strongly rainbow connected.

The following table and figure1 illustrates the Strongly Rainbow connected path in $G_S(Z_7)$

It is clear that $rc(G_S(Z_7))=src(G_S(Z_7))=2$

Path: $x \rightarrow y$	$x \rightarrow z \rightarrow y$	$d(x, y)$	Length of Rainbow u-v path
$1 \rightarrow 6$	$1 \rightarrow 5 \rightarrow 6$	2	2
$2 \rightarrow 5$	$2 \rightarrow 4 \rightarrow 5$	2	2
$3 \rightarrow 4$	$3 \rightarrow 6 \rightarrow 4$	2	2

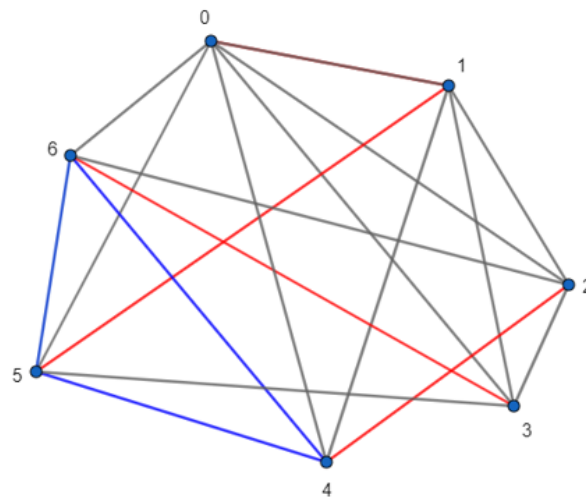


Figure 1

Lemma: In (Z_n, \times) where n is prime, $(n-1)$ is always self-invertible since $((n-1))^2=1$,

Where 'n' is other than the identity

Proof: $(n-1)$ is the self-invertible element

$\therefore \exists$ two self invertible elements namely 1, $(n-1)$ in every (Z_n, \times) where n is prime since $((n-1))^2=1$

i.e, $((n-1))^2 \equiv 1 \pmod{n}$ [4]

$\Rightarrow ((n-1))^{-1} = (n-1) \quad \forall$ prime n

Theorem: For $x, y \in S$ and $x^{-1} \neq y$, x and y are not adjacent iff $xy = (n-1)$ where n is prime

Proof: Let $x, y \in S \Rightarrow x, y$ are non-self-invertible

Necessary condition: Consider that $x^{-1} = y$ and x, y are non-adjacent then $xy \notin S$

As $[Z]_{n-S}$ consists of only $1, (n-1)$ which are self-invertible elements

$\Rightarrow x, y$ must be equal to $(n-1)$, since $x^{-1} \neq y$ so $xy \neq 1$

Sufficient condition: Let $x, y \in S$ and $xy = (n-1)$

$\Rightarrow x^{-1} \neq y$ so $xy \neq 1$

\therefore By definition of Inverse graph, as $xy = (n-1)$ and $x^{-1} \neq y$ and $(n-1) \notin S$

$\Rightarrow x$ and y are not adjacent.

Theorem: The Inverse graph $G_S(Z_n)$, $\forall n$ (n is prime) corresponding to the cyclic group

(Z_n, \times) is Rainbow connected and hence is Strongly Rainbow connected such

that $\text{rc}(G_S(Z_n)) = \text{src}(G_S(Z_n)) = 2$

Proof: Let (Z_n, \times) be a cyclic group iff n is prime

Let S be a non-empty subset of non-self-invertible elements of Z_n and $G_S(Z_n)$,

be the corresponding inverse graph of Z_n

Case 1 : Adjacency between the identity element and every element of S

The identity element '1' in (Z_n, \times) is adjacent to all the elements of S and $d(1, v) = 1 \forall v \in S$

Case 2 : Adjacency between the elements of S

There arise 2 sub cases

Subcase 1: For any $x, y \in S$, if $xy \in S \Rightarrow x, y$ are adjacent

Subcase 2: For any $x, y \in S$, if $xy \notin S$, there arise the following observations.

$xy = 1$, for $x \neq y$

Then in the inverse graph $G_S(Z_n)$, x, y are non-adjacent as $x^{-1} = y$

\exists an element $z \in Z_n - S$ ($z = 0$ or $(n-1)$) such that $xz \in S$ and $yz \in S$

$\Rightarrow x, z$ are adjacent in $G_S(Z_n)$ as well as z, y are adjacent in $G_S(Z_n)$

$\Rightarrow \exists$ a path between x and y namely $x \rightarrow z \rightarrow y$

$d(x, y) =$ Minimum length of rainbow connected path between x and y is equal to 2

Hence there exists a rainbow connected path between every pair of elements x, y in S , for $xy = 1$

$xy = (n-1)$, x is neither the inverse of y nor adjacent to y

$\forall x \in Z_n, \exists y (\neq x) \in Z_n$ such that $xy = (n-1)$ and $x \neq y^{-1}$

\Rightarrow In the inverse graph $G_s(Z_n)$, x, y are non-adjacent

\exists an element $z \in Z_n - S$ ($z=0$ or $(n-1)$) such that $xz \in S$ and $yz \in S$

$\Rightarrow x, z$ are adjacent in $G_s(Z_n)$ as well as z, y are adjacent in $G_s(Z_n)$

$\Rightarrow \exists$ a path between x and y namely $x \rightarrow z \rightarrow y$

$d(x, y) =$ Minimum length of rainbow connected path between x and y is equal to 2

Hence, there exists a rainbow connected path between every pair of elements x, y in S , for $xy = (n-1)$

Case 3: Adjacency between two self-invertible elements

\exists Exactly two self-invertible elements $1, (n-1) \in Z_n - S$

Then there does not exist adjacency between 1 and $(n-1)$

\therefore There must exist an element say $z \in S$ such that \exists a path $1 \rightarrow z \rightarrow (n-1)$

$\therefore d(1, (n-1)) = 2$

Hence there exists a rainbow connected path between every pair of elements x, y

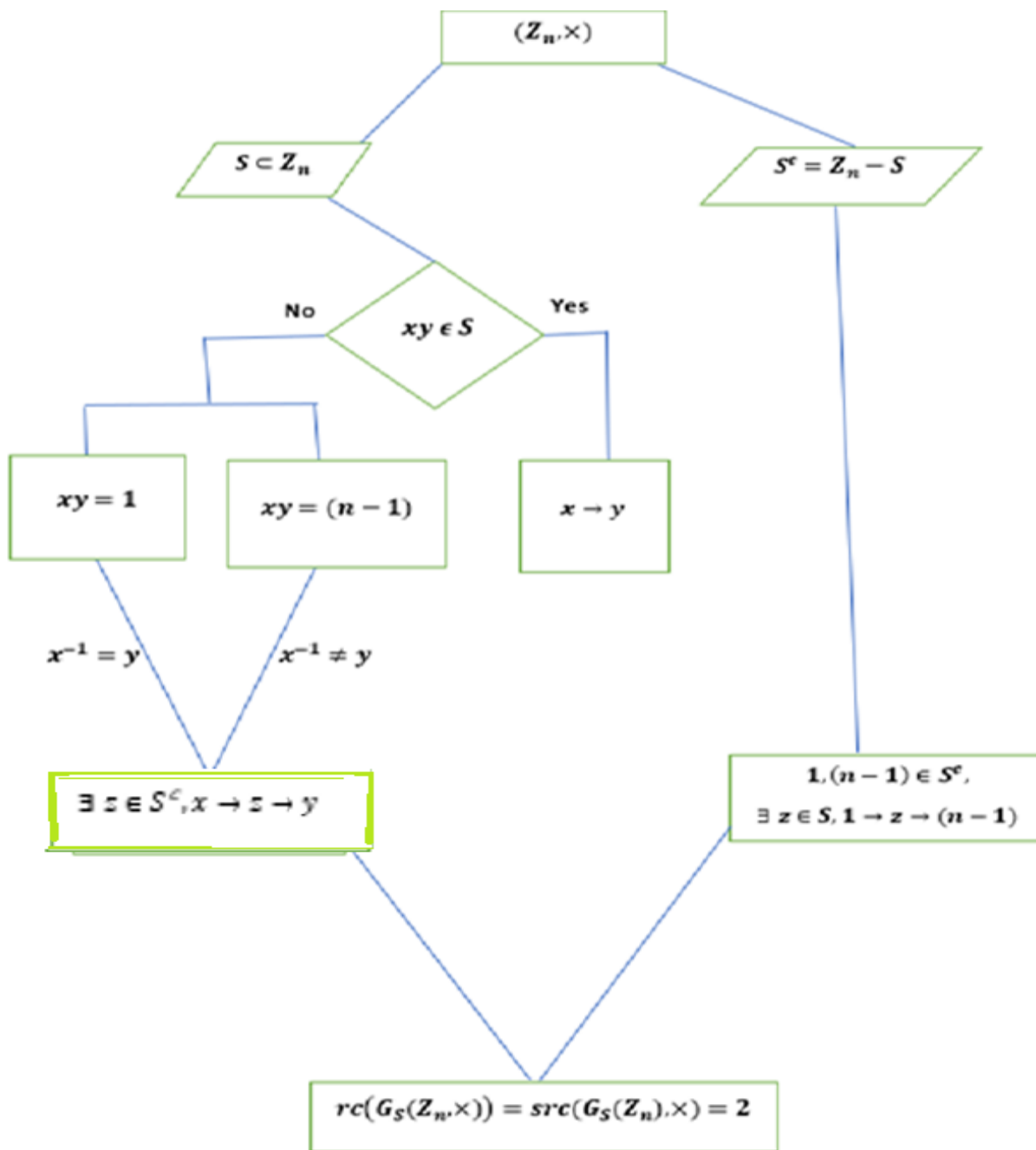
in $Z_n - S$

By considering all the 3 cases, $G_S(Z_n)$ is Rainbow connected

x, y are geodesic and $G_S(Z_n)$ is strongly rainbow connected, $\forall x, y \in [Z_n]$

$\therefore rc(G_S(Z_n)) = src(G_S(Z_n)) = 2$

The following algorithm illustrates Rainbow connection path of the inverse graph $(G_S(Z_n), \times)$



Let us illustrate the above theorem by the following example.

In the Inverse graph $G_S(Z_7)$ corresponding to the cyclic group (Z_7, \times) is strongly rainbow connected.

The following table and figure2 illustrates the Strongly Rainbow connected path in $G_S(Z_7)$

It is clear that $rc(G_S(Z_7)) = src(G_S(Z_7)) = 2$

Path: $x \rightarrow y$	$x \rightarrow z \rightarrow y$	$d(x, y)$	Length of Rainbow u-v path
$1 \rightarrow 6$	$1 \rightarrow 2 \rightarrow 6$	2	2
$2 \rightarrow 3$	$2 \rightarrow 6 \rightarrow 3$	2	2
$2 \rightarrow 4$	$2 \rightarrow 1 \rightarrow 4$	2	2
$3 \rightarrow 5$	$3 \rightarrow 1 \rightarrow 5$	2	2
$4 \rightarrow 5$	$4 \rightarrow 6 \rightarrow 5$	2	2

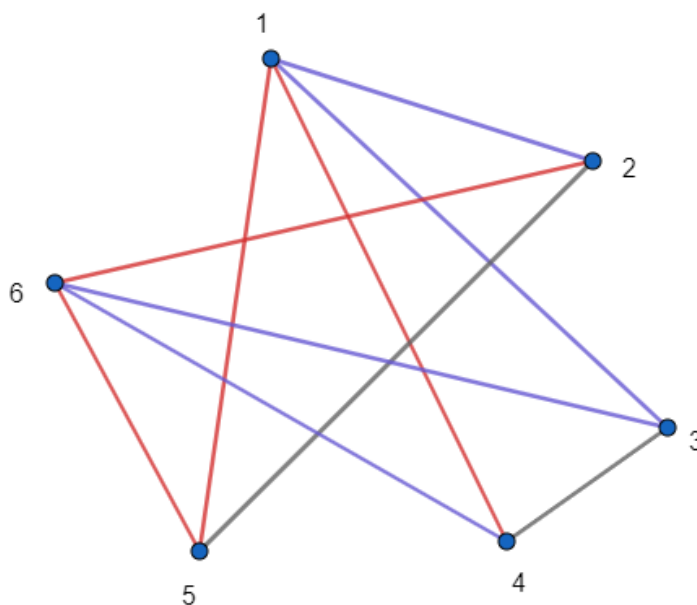


Figure 2

Conclusion:

The main conclusion of this study is to investigate the Strongly Rainbow Connected number of the Inverse graph $G_S(\mathbb{Z}_n)$ corresponding to the cyclic group $(\mathbb{Z}_n, +)$ and (\mathbb{Z}_n, \times) by employing the techniques and concepts of Number Theory, by establishing the lemma that, $xy = (n-1)$ where n is prime iff x and y are not adjacent, for $x, y \in S$ and $x^{-1} \neq y$. Further the critical Property of Rainbow connected graph of the corresponding inverse graph has been investigated and proved that $rc(G_S(\mathbb{Z}_n)) = \text{src}(G_S(\mathbb{Z}_n)) = 2$

REFERENCES

- [1] Ambika B.J., Naga Maruthi Kumari N., Banga M.K. A Survey on the Path Restoration in MPLS Networks. In: Guru D., Vasudev T., Chethan H., Kumar Y. (eds) Proceedings of International Conference on Cognition and Recognition. Lecture Notes in Networks and Systems, vol 14. Springer, Singapore. (2018).
- [2] D.F. Anderson., M.C. Axtell and J.A. Stickles., Zero-divisor graphs in commutative rings, in Commutative Algebra, Noetherian and non-Noetherian

- Perspectives (M. Fontana, S. Eabbaj, B. Olberding, I. Swanson, Eds), Springer-Verlag, New York, 2. 11, 23–45, (2010).
- [3] Gary Chartrand., Kalamazoo., Garry L. Johns., Rainbow Connection in Graphs, *Mathematica Bohemica.*, No. 1, 85–98,133(2008)
- [4] G.H.Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University press, Sixth edition (2008),ISBN:978-0-19-921985-8
- [5] K. Srinivasa Rao., U Vijaya Chandra Kumar., Anjaneyulu Mekala., Rainbow Connection Number of Flower Snark graph, *International Journal of Applied Mathematics.*, Volume 33 No. 4 591-597, 2020
- [6] M Gopalakrishnan., NNM Kumari., Generator graphs for cyclic groups, *AIP Conference Proceedings*, 2019
- [7] M. Lakshmi Kameswari., N. Naga Maruthi Kumari and T. V. Pradeep Kumar., A graphical Approach to groups and its properties, *International Journal of Research in Engineering, IT and Social Sciences*, Volume 9, Issue 04, April 2019
- [8] Monther R. Alfuraidan, Yusuf F. Zakariya, Inverse graphs associated with finite groups, *Electronic Journal of Graph Theory and Applications* 5(1),142-154, (2017).
- [9] Sivakumar Pathuri ., N. Naga Maruthi Kumari., Isomorphic graphs based on isomorphic groups, *AIP Conference Proceedings* 2112, 020127 (2019)
- [10] W.B Vasantha Kandasamy., Florentin Smarandache., *Groups as Graphs*, Era CdituuArt (2009)ISBN-13: 978-159-97309-3-6