Original Article

Variational Iteration Method for Solution of Two Dimension Fractional Diffusion Equations

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DOI: 10.47750/pnr.2022.13.S01.177

Abstract

In the current research work we have introduced the Variational Iteration Method to obtain the Analytical Approximate Solution of the Two-Dimensional Diffusion Equation. We have used the initial value to determine the solution for various mathematical problems. This in turn has helped in accelerating the rapid convergence of the solution for the series. We have used the Variational Iteration method to solve the fractional-order diffusion equations for two dimensions. The series form solution for the fractional -order diffusion problem is obtained by the proposed method. We have effectively identified solutions for a wide range of physical and analytical problems during our research. Since we have demonstrated its effectiveness in finding the solutions for the fractional diffusion equations of two dimension, we can now say that VIM can be extremely helpful in tackling a wide range of engineering complexities. The Variation Iteration method has time and again proved to give exemplary result in terms of efficacy and simplicity. It has been proved superior and advantageous over the previously used analytical techniques Adomian Method, New Homotopy Perturbation Method, Homotopy [7.11,12] etc. The primary benefit of this strategy is that it allows us to achieve the desired result in a short amount of time. The obtained results are incredibly accurate and effective. We can also adapt this method for various other non-linear mathematical and scientific problems. We have tried to project mathematical results for various problems in the current research paper.

Keywords: Variational Iteration Method, Two Dimension; Fractional Diffusion Equation; Analytical Solution.

INTRODUCTION

The fractional calculus is widely accepted and used to solve problems at various levels of sciences and engineering. Researchers such as Oldham and Spanier [1] were pioneer in the development of the method. Other fellow researchers such as Miller [2], Diethelm and Ford [3], Diethelm [4], and have contributed immensely in solving the fractional differential equations. It is time and again proved that the Variational Iteration Method is a powerful tool to obtain exact solutions for nonlinear equations; and hence this method is highly accepted among fellow re-searchers and community. Nonlinear PDF and nonlinear differential equations of the fractional order has been successfully solved by using the Variational Iteration Method. This method was first introduced by He [5, 6]. Homotopy Perturbation Method applied by Fatima & Danial [12] to solve heat and wave equations and new HPM also applied by Fatima & Dhariwal [11] to solve heat and Laplace equation.

Analytical fractional diffusion is represented in time by the equation below:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right), \quad 0 < \alpha \leq 1, D > 0$$

$\frac{\partial^\alpha}{\partial t^\alpha}(.)$ represents the Caputo derivative of order, the probability density function to find a particle at x in time t is being represented by $\alpha, u(x,t)$ There are various factors on which the positive constant D is dependent on, such as the universal gas constant, temperature, the friction coefficient and finally on the external force Avagadro number F(x)

Let us consider,

$D = 1, \alpha = \frac{1}{2}$ and $F(x) = -x.$

In order to resolve the above issue, researchers Saha Ray and Bera [9] utilized the Adomian Decomposition Method [ADM]. Since the procedure for calculating the Adomian polynomials is very complex and time challenging, it is therefore not usually accepted by various experts in the field. Hence this has fundamental disadvantage when we use this method. Das [9], [12] and Fatima [10] successfully applied variational iteration method to solve nonlinear systems of PDE's and nonlinear differential equations of fractional order. In order to overcome the drawbacks of the Adomian method, we used the Variation Iteration Method (often referred to as VIM). For diverse values of x and t in specific conditions, the analytical equation of u. (x, t) is being developed and we have graphically represented it by using initial conditions. The most sophisticated feature of this method is its straight forward approach it takes to find a conclusive solution to a given condition.

Problem Solution

Let us take the following equation;

$$\frac{\partial^{\frac{1}{2}} u(x,y,t)}{\partial t^{1/2}} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{\partial}{\partial x}\left(xu(x,y,t)\right) + \frac{\partial}{\partial y}\left(yu(x,y,t)\right)$$

with initial condition *u(x,y, 0) = f(x,y).*

For the two-dimensional Variation Iteration Method (VIM), the correction function in t-direction is being considered in the below

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,y,\xi)}{\partial \xi} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2 u_n(x,y,\xi)}{\partial x^2} + \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2 u_n(x,y,\xi)}{\partial y^2} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial}{\partial x}\left(xu_n(x,y,\xi)\right) - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial y}\left(yu_n(x,y,\xi)\right)\right] d\xi$$

By calculating the Lagrange multiplier λ, the successive approximation $u_j, j \geq 0$ can be established. If $\delta u_n = 0$, then u_n is a restricted variation. By using any selective function u_0, the successive approximation $u_{n+1}(x,y,t), n \geq 0$ of the solution *u(x, y, t)* can be derived easily by applying Lagrange's multiplier. For the zeroth approximation of u_0, the initial value *u(x, y,0)* and $u_t(x, y, 0)$ is being used..

For equating the value of λ,

$$\delta u_{n+1}(x,t) = \delta u_n(x,y,t) + \delta \int_0^t \lambda(\xi) \frac{\partial u_n(x,y,\xi)}{\partial \xi} \partial \xi = 0$$

This will give the below fixed condition

$\lambda'(\xi) = 0$

and $1 + \lambda(\xi) = 0$

which gives $\lambda = -1$

Finally, the exact solution is obtained by

$$u(x,y,t) = \lim_{n \to \infty} u_n(x,y,t)$$

To put it another way, the correction functional (4) will yield numerous approximations, and the actual solution can be found.

To prove this method some examples are shown here.

Illustrative Examples

Example 1

We take 2-dimensional fraction order equation given in Shah et. al. (8)

$$\frac{\partial^{1/2}u}{\partial t^{1/2}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

with the initial condition,

$$u(x, y, 0) = (1 - y)e^x$$

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial^2}{\partial x^2} u(x,t) + \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial^2}{\partial y^2} u(x,t)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} u_n(x,\xi) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} u_n(x,\xi)\right] d\xi$$

$$\lambda = -1$$

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} u_n(x,\xi) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} u_n(x,\xi)\right] d\xi$$

For n = 0

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} u_0(x,\xi) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} u_0(x,\xi)\right] d\xi$$

$$= (1-y)e^x - \int_0^t \left[0 - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} (1-y)e^x - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} (1-y)e^x\right] d\xi$$

$$= (1-y)e^x + \int_0^t \left[2\sqrt{\frac{t}{\pi}}(1-y)e^x + 0\right] d\xi$$

$$u_1 = (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}}\right]$$

For n = 1

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\xi)}{\partial \xi} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} u_1(x,\xi) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} u_1(x,\xi)\right] d\xi$$

$$= (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}}\right] - \int_0^t \left[\frac{\partial}{\partial \xi}(1-y)e^x \left(1 + 2\sqrt{\frac{t}{\pi}}\right) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} u_1\right] d\xi$$

$$= (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}}\right] - \int_0^t \left[(1-y)e^x \frac{1}{\sqrt{t\pi}} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} (1-y)e^x \left(1 + 2\sqrt{\frac{t}{\pi}}\right)\right] d\xi$$

$$u_2 = (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}} + t\right]$$

$$u_3 = (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}} + t + \frac{4}{3}t\sqrt{\frac{t}{\pi}}\right]$$

Hence, the solution is

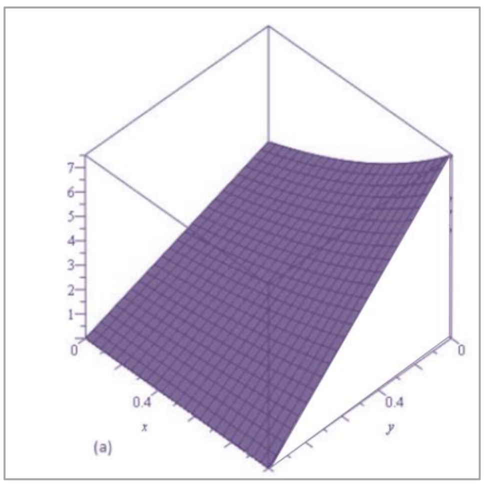
$$u = u_1 + u_2 + u_3 + u_4 + \cdots$$

$$= (1-y)e^x \left[1 + 2\sqrt{\frac{t}{\pi}} + t + \frac{4}{3}t\sqrt{\frac{t}{\pi}} + \cdots..\right]$$

The exact answer is determined in a closed form:

$$u = (1-y)e^{x+t}$$

Fig 1. Solution of Example 1 and $0 < x \leq 1$ from [9]



The above Figure represent a three-dimensional surface for 2-dimensional fraction order equation given in Shah et. al. (8*)* and obtain *u(x, t)* for different values of t. Both precise and VIM answers are in strong agreement with each other, as can be seen from the graphs from [9].

Example 2

Let us consider the fractional diffusion equation in two dimensions.

$$\frac{\partial^{\gamma} v}{\partial t_1^{\gamma}} = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial y_1^2}, \qquad 0 < \gamma \leq 1, \qquad t_1 \geq 0,$$

with the starting state;

$$u(x, y, 0) = e^{x+y}$$

$$u_1 = e^{x+y} - \int_0^t \left[0 - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial x^2} e^{x+y} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \frac{\partial^2}{\partial y^2} e^{x+y} \right] d\xi$$

$$u_1 = e^{x+y} + \int_0^t \left[\frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} e^{x+y} + \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} e^{x+y} \right] d\xi$$

$$u_1 = e^{x+y} + e^{x+y}\, 2\sqrt{\frac{t}{\pi}}$$

$$u_1 = e^{x+y} \left(1 + 2\sqrt{\frac{t}{\pi}} \right)$$

$$u_2 = e^{x+y} \left(1 + 2\sqrt{\frac{t}{\pi}} + 4t \right)$$

$$u_3 = e^{x+y} \left(1 + 2\sqrt{\frac{t}{\pi}} + 4t + \frac{32t}{3}\sqrt{\frac{t}{\pi}} \right)$$

$$u = u_1 + u_2 + u_3 + u_4 + \cdots$$

$$u = e^{x+y} \left(1 + 2\sqrt{\frac{t}{\pi}} + 4t + \frac{32t}{3}\sqrt{\frac{t}{\pi}} + \cdots \right)$$

The precise answer is computed in a closed form:

Example 3

$$\frac{\partial^{\frac{1}{2}} u}{\partial t^{1/2}} = \frac{\partial^2}{\partial x^2} (u(x, y, t)) + \frac{\partial^2}{\partial y^2} (u\{x, y, t\}) + \left\{ \frac{\partial}{\partial x} (xu\{x, y, t)) + \frac{\partial}{\partial y} (yu\{x, y, t) \right\}$$

with initial condition $u_0 = x$

$$u_1 = x + \int_0^t \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left[\frac{\partial^2}{\partial x^2} (u_0) + \frac{\partial^2}{\partial y^2} (u_0) + \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left\{\frac{\partial}{\partial x} (xu_0) + \frac{\partial}{\partial y} (yu_0)\right\}\right]$$

$$u_1 = x + \int_0^t \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left[0 + 0 + \frac{\partial^{1/2}}{\partial \xi^{1/2}} (3x)\right] d\xi$$

$$u_1 = x + 3x \int_0^t \frac{\partial^{1/2}}{\partial \xi^{1/2}} [1] d\xi$$

$$u_1 = x + 3x \int_0^t \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{1}{\sqrt{t\pi}} d\xi$$

$$u_1 = x + 6x \sqrt{\frac{t}{\pi}}$$

$$u_2 = x + 6x \sqrt{\frac{t}{\pi}} \int_0^t \frac{\partial}{\partial \xi} \left(x + 6x\sqrt{\frac{t}{\pi}}\right) - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left[\frac{\partial^2}{\partial x^2} \left(x + 6x\sqrt{\frac{t}{\pi}}\right) + \frac{\partial^2}{\partial x^2} \left(x + 6x\sqrt{\frac{t}{\pi}}\right)\right]$$
$$- \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left[\frac{\partial}{\partial x} \left(x^2 + 6x^2\sqrt{\frac{t}{\pi}}\right) + \frac{\partial}{\partial y} \left(y + 6x\sqrt{\frac{t}{\pi}}\right)\right]$$

$$u_2 = x + 6x \sqrt{\frac{t}{\pi} + 9xt}$$

$$u_3 = u_2 + \int_0^t \frac{\partial u_2}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \left[\frac{\partial^2}{\partial x^2} (u_2) + \frac{\partial^2}{\partial y^2} (u_2) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \left\{\frac{\partial}{\partial x} (xu_2) + \frac{\partial^2}{\partial y^2} (u_2)\right\} - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \left\{\frac{\partial}{\partial x} (xu_2) + \frac{d}{\partial y} (yu_2)\right\}\right]$$

$$= x + 6x\sqrt{\frac{t}{\pi}} + 9xt - \int_0^t \left[\frac{1}{\sqrt{tx}} + 9x - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} (0+0) - \frac{\partial^{\frac{1}{2}}}{\partial \xi^{\frac{1}{2}}} \left(2x + 12x\sqrt{\frac{t}{\pi}} + 18xt + x + 6x\sqrt{\frac{t}{\pi}} + 9xt\right)\right]$$

$$u_3 = x + 6x\sqrt{\frac{t}{\pi}} + 9xt + \frac{36xt^{3/2}}{\sqrt{\pi}}$$

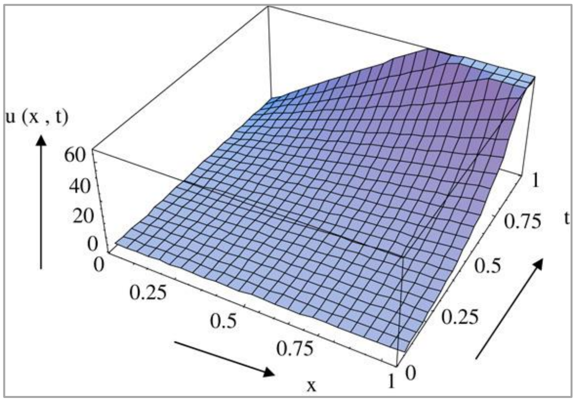
Now the solution;

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$= x + 6x\sqrt{\frac{t}{\pi}} + 9xt + \frac{36xt^{3/2}}{\sqrt{\pi}} + \cdots$$

The preceding finding agrees completely with Saha Ray and Bera.

Fig 2. Solution of Example 3 and $0 < x \leq 1$ from [9]



NUMERICAL RESULTS AND DISCUSSION

Figure 2 from [9] represent a three dimensional surface for u(x, t) w.r.t x & t and for u(x, t) for different values of t at $x = 1$. The graphic demonstrates that in both instances, u(x, t) increases as x and t increase. Same as in Figure 1 from [9] u(x, t) also increases with different t values.

CONCLUSION

To arrive at an analytical solution to the fractional diffusion problem, we've used the Variational Iteration Method (VIM). We have effectively identified solutions for a wide range of physical and analytical problems. Since we have demonstrated its effectiveness in finding the solutions for the fractional diffusion equations, we can now come to a conclusion that VIM can be extremely helpful in tackling a wide range of engineering concerns.

The foremost advantage of this method is that we can get to the desired result in quick timeframe. The results which are obtained are extremely close and effective. Most importantly, when we use the Adomian decomposition method to come to a conclusion, it bypasses lengthy calculations which is required by Adomian polynomials.

The results achieved in the current research work had a mutual understanding amongst the researchers. A mathematical model for any experimental data may now be created using several kinds of fractional orders. This way we can arrive at a conclusive solution.

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ACKNOWLEDGMENT

The authors wish to thank Prince Sultan University Riyadh for their support.

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